To His Excellency The Prince Boncompagni
With the author's high Respect

Florian Cajori

ON

THE GENERAL

PRINCIPLES OF ANALYSIS;

BEING

A Series of Original Investigations

IN

DIFFERENT DEPARTMENTS OF PURE MATHEMATICS.

PART I.

THE ANALYSIS OF NUMERICAL EQUATIONS.

BY

J. R. YOUNG,

LATE PROFESSOR OF MATHEMATICS IN BELFAST COLLEGE.

LONDON:

C. C. SPILLER, 102, HOLBORN HILL;

SAMUEL MAYNARD, MATHEMATICAL BOOKSELLER,

EARL'S COURT, LEICESTER SQUARE, LONDON.

1850.

4

PRICE TWO SHILLINGS AND SIXPENCE.

NOTICE.

The work, of which this is the First Part, was intended to be issued in monthly portions, and to embrace the following topics: 1. Conjugate Functions, and their application to the Analysis of Equations. 2. Researches respecting the Theory of Numbers. 3. A Disquisition on Diverging Series. 4. On Zeros, real and imaginary, Conjugate Points, and Imaginary Logarithms. 5. On certain Theorems of Leibnitz, Abel, Murphy, &c. 6. On the Law of Continuity in Analysis. 7. Miscellaneous Investigations in the Calculus of Operations, the Theory of Probability, the Law of Population, &c. 8. Prefatory matter, in a Dedication to the Earl of Clarendon, containing statements and explanations of interest to the friends of Science.

The author feels it necessary here to state, that the further prosecution of this design must, for the present, be suspended. The subscribers to the work, although including some of the most distinguished names in the kingdom, are nevertheless too few in number, as yet, to justify him in incurring the expense of the publication. The proceeds of this First Part of the work are not likely to cover one half of the outlay: as soon as this obstacle is removed, the publication will be regularly proceeded with.

The Statements, &c., intended for Part VIII will shortly be printed in a distinct form. The pamphlet will be published separately at One Shilling: the subscribers to the work originally proposed will not be considered as pledged to the purchase of it, nor will it be forwarded to them unless expressly ordered.

One remark more must be made, though it would be gladly avoided. It was the early-formed resolution of the author that duplicate copies should not be forwarded to any individual subscriber; this resolution was founded upon reasons cogent to his own mind, but which he may, perhaps, here be excused from formally stating. Some of the most eminent persons whose names occur in the following list, have each ordered several copies of the publisher. But the author respectfully requests permission to adhere to his original determination, though at the risk of incurring the charge of inconsistency. His hope and aim was, and is, that his book might, in some degree, be acceptable to science; and he must be suffered to indulge the delusion —if delusion it be—that the countenance extended to it is justified by the expectation of at least partial success in his endeavours, irrespective of all considerations personal to the author.

ON THE

GENERAL PRINCIPLES OF ANALYSIS.

INTRODUCTION.

The present work will consist of a series of independent essays upon some points of interest in analytical mathematics. And although the several subjects to be discussed will naturally fall under special titles, I shall endeavour to blend with the examination of them, such occasional remarks and observations as may seem to me to throw some light upon the management and texture of algebraic operations in general; and to bear, more or less closely, upon the entire science of symbolical reasoning. It is on this latter account that I have ventured to call the work, 'A Treatise on the General Principles of Analysis.' The title certainly appears an ambitious one:—I fear it may not be justified by the execution. But if those who take a pleasure in abstract researches, should think that I have succeeded, now and then, in illustrating a previous obscurity, or in adding a new truth, or a new principle, to the existing stock, however trifling such additions may be, *they*, at least, will not, I am persuaded, be too severely critical upon so unimportant a matter.

In prosecuting the work, I have not kept any particular class of readers before my mind. I have certainly assumed that some general acquaintance with the present state of the topics of

inquiry introduced has been already acquired; but, as I have aimed to be intelligible and perspicuous, I think the student of general analysis, whose researches have not extended beyond the limits of our ordinary elementary books, may read the volume with ease and, I hope, with advantage.

It seems to me to be a libel upon mathematical science to pronounce any of its departments to be inherently too difficult and recondite for the comprehension of a mind thus trained and prepared. Analysis is too frequently and too unjustly branded with this stigma, when the reproach more properly belongs to those who are the exponents of its doctrines. If superior clearness and perspicuity can reasonably be looked for in any subject, surely that subject is mathematics: and yet no department of scientific research can be adduced which is enveloped in so much unnatural and repulsive obscurity as the science of abstract analysis. Still, however, I must acknowledge, that this fault is not always without its advantages; what is of easy attainment is often undervalued; and, in the minds of some, the only way to enhance the estimate is to increase the difficulty of access. It was, no doubt, a conviction of this truth, that influenced Mr. HORNER to invest his important discovery of the general solution of numerical equations, in the dress in which it appears in the 'Transactions of the Royal Society.' With admirable tact and sagacity, he shrouded a very simple matter in a garb of transcendental mystery; but, as soon as his primary object was secured, the object, namely, of the early publication of his results, under the auspices of a learned body, he proceeded to strip the subject of its unnatural disguise, and to exhibit it in all its native simplicity.

From these introductory remarks, it will be perceived, that my object will be to convey the contents of this volume to the reader with as little expenditure of thought and labour, on his part, as he can reasonably expect to incur in the perusal.

I do not see why a principle, universally adopted in our manual operations, and in our industrial economy—the minimum of exertion to the maximum of result—should not be equally applicable to our intellectual processes; nor why the muscles of the

mind should be exercised for nothing, any more than the muscles of the body. Labour—notwithstanding the old motto—is surely not desirable for its own sake. The youthful powers, whether of mind or body, unquestionably require to be stimulated to exertion, in order to their full and healthy development, though nothing else be gained by the effort. But too much discipline may be as bad as too little; a time should arrive when the apprenticeship ought to expire, and the novice be pronounced qualified to set up for himself.

I should consider these latter observations as quite superfluous, but for the opinions recently promulgated by a writer of great and deserved eminence. It seems to be the conviction of Dr. WHEWELL that, till two- or three-and-twenty, the scientific course of a young man should be exclusively one of intellectual discipline. Invigorating mental exercise, and not substantial mental attainment, is to be the object kept in view; and if two paths lead equally to the same end, the one long and laborious, and the other short and easy, the former is to be chosen instead of the latter. It is a necessary consequence of this doctrine, that analysis is to be displaced for geometry. But I cannot conceive upon what grounds Dr. WHEWELL denies to analysis the same logical rigour and conclusiveness which he justly claims for geometry. I should say that, if any department of inquiry be destitute of these, it should be at once expunged from the category of mathematics. The only difference between analytical and geometrical reasoning is in the language used—the instrument employed. I have no conception of two kinds of *reasoning*, essentially and inherently different. All reasoning must, in its characteristic features, be the same; the subjects may be different, and the apparatus may be different; but there can be no difference in the mental process. If the conclusiveness of any step in a train of algebraic deduction be questioned, the analyst would justify it exactly as the geometer would justify a geometrical conclusion: that is, by showing that it necessarily and syllogistically follows from what has been already admitted. In both methods the mind is exercised in precisely the same way, however different the visible symbols of its operations.

In the writings of geometers, as well as in those of analysts, errors and paralogisms may be pointed out; and it must be confessed that, in analysis, the temptations to go astray often assume a very seducing aspect, and have often prevailed. But this admission rather tells against, than for, Dr. WHEWELL's argument; it is an admission that, in analysis, caution, circumspection, and sagacity are pre-eminently demanded, and require to be constantly exercised.

SECTION I.

THE ANALYSIS OF NUMERICAL EQUATIONS.

(1.) From the very partial success attendant upon all the methods of late years proposed for the analysis of numerical equations, I am inclined to think that, if any considerable advance is to be made in this important subject, it must be made by pursuing the path along which NEWTON took the initial step, availing ourselves of the light that the researches of DESCARTES have thrown over it.

It was in this direction that BUDAN prosecuted the inquiry. His method is usually called "an extension of the rule of DESCARTES." But whatever aid this valuable rule afforded, and however indispensable that aid in BUDAN's process, the process itself seems to me to be just that which, in the natural order of discovery, would be expected to follow that of NEWTON, regarded as a first step in a clearly indicated direction.

(2.) A numerical equation is said to be analysed as soon as we discover the several *limits*, or pairs of numbers, within which all its unequal real roots lie individually, and its equal roots in distinct groups; that is, as soon as these unequal roots, and

groups of equal roots, are all separated and severally enclosed, each between two assignable numbers.

The preliminary step in this complete analysis of an equation would seem to be, first to determine limits as close as practicable, within which *all* the roots should be comprised, and this was NEWTON'S step. He showed how close superior and inferior limits were to be found without the bounds of which no roots could possibly exist. BUDAN cut up NEWTON'S wide interval into a set of partial intervals, and showed that some of these partial intervals might, in like manner, be rejected; and that the real roots were to be found only among the intervals which he retained.

The theorem of BUDAN, by which this further advance in the analysis of an equation is effected, was first published in 1807. The same theorem is, however, claimed by the friends of FOURIER, more especially by NAVIER, who published the posthumous work on Equations, by the latter distinguished analyst, in 1831; and in which this theorem is given without any reference to BUDAN'S printed researches; there is certainly every reason to suppose that FOURIER was led to it independently; but as priority of publication unquestionably belongs to BUDAN, and as no documentary evidence has been produced in favour of FOURIER'S still earlier claim, I shall here, as in my former work, continue to call the theorem in question the 'Theorem of BUDAN;'* regarding this writer as the first who clearly marked out for rejection certain portions of NEWTON'S interval, within which all search for roots would be fruitless.

(3.) In the investigations which follow, I attempt to advance a step further in the same direction; and as BUDAN pointed out rejective intervals within the bounds of the extreme limits of the roots, so I seek for rejective intervals among those which he has retained.

And here I would, at the outset, remark that, in an equation of

* Some discussion, respecting FOURIER'S independent title to this theorem, will be found in my treatise on the 'Theory of Equations,' p. 150, and in the Preface to that work.

an advanced degree, it requires, in general, no small amount of trouble to obtain all BUDAN's subdivisions of the interval; and often very considerable labour to ascertain the true character of the roots of which these subdivisions contain the indications. In writings on equations, much of this necessary work is usually suppressed: the rows of signs, due to the limits of BUDAN's partial intervals, are all exhibited to the eye; but the process by which they are obtained is usually omitted. Indeed, throughout the whole of this business of analysing an equation, preparatory to actual solution, there is frequently, as every practical analyst knows, a great deal of arithmetical work of which no trace appears in the finished process.

In numerical operations, few things are more dispiriting than to find, after a long course of calculation has been carefully gone through, that the whole might have been spared, from it ultimately appearing that we have been unconsciously prosecuting researches in a direction in which the object sought cannot possibly exist.

This is the sort of disappointment which we must often meet with in discussing the intervals within which pairs of roots are indicated by BUDAN's rows of signs. And I think that any easily-understood and easily-applied directions, which can unerringly warn us off any of these unproductive paths, would be, in some measure, acceptable to science.

(4.) In the present preliminary observations, I have not adverted to the theorem of STURM, because, although this theorem is theoretically perfect, yet, as ample experience has shown, it is impracticable beyond very narrow limits. No one, I think, will be inclined to repeat the labour which I have undertaken, in analysing an equation of the sixth degree, with moderately large coefficients, by this theorem; the calculations involving products of more than a hundred and fifty figures in a row.* It will always, however, be regarded as one of the most beautiful discoveries in this department of Analysis, and must ever command

* Theory and Solution of Equations of the Higher Orders; 1843, p. 277.

a prominent place in every complete exposition of the general theory of numerical equations.

(5.) The method now to be explained is derived from a discussion of the identical equation,

$$X=\{F+\sqrt{F^2-X}\}\times\{F-\sqrt{F^2-X}\}\ldots\ldots\ldots(\mathrm{I})$$

in which X and F may be any functions whatever; so that if one of these be fixed, or prescribed, the other is quite arbitrary.

This identity, as will be at once seen, is nothing but an immediate inference from the simple algebraic truth, that the sum of any two quantities, multiplied by their difference, furnishes the difference of the squares of those quantities. But the only instances I am acquainted with, in which even any particular case of the general principle (I) has been employed in analysis, are two in number; one such particular application will be found in Mr. GOMPERTZ's 'First Tract on Imaginary Quantities,' and the other in a Paper by Mr. COCKLE, in the 'Philosophical Magazine' for February, 1849. Besides these, I know of no other instances of the introduction of an arbitrary function into a pair of *conjugate factors;* and I believe the general form (I), notwithstanding its axiomatic simplicity, had escaped notice till the publication of it by myself in the Magazine for April, 1849.

(6.) Since F in the formula (I) is entirely arbitrary, though X be fixed, we are at liberty, in any particular case of (I), to write afterwards $F+f$ for F, where f is equally arbitrary; that is, we may always change (I) into

$$X=\{F+f+\sqrt{F^2+2Ff+f^2-X}\}\times\{F+f-\sqrt{F^2+2Ff+f^2-X}\}\ldots(\mathrm{II})$$

from which we see, that having decomposed any function X into a pair of conjugate factors, we may always afterwards add any quantity f to the rational part of each factor, provided we, at the same time, introduce the expression $2Ff+f^2$ under the radical sign; and it was the recognition of this privilege, that suggested the following course of inquiry.

(7.) Before proceeding to more general forms, let us first consider the quadratic equation

$$x^2 + px + q = 0 \dots\dots\dots\dots (1)$$

the first member of which, when written thus,

$$x^2 - (-px - q)$$

immediately suggests its simplest pair of conjugate factors, namely,

$$x + \sqrt{(-px - q)}$$

$$x - \sqrt{(-px - q)}$$

which, x being put for the arbitrary quantity F, and x^2+px+q for X, agree with the form (I).

If, now, we introduce f as in (II), and determine it so that $2f$ may be equal to p, that is, make f equal to $\frac{1}{2}p$, x, under the radical, will disappear; and another pair of conjugate factors, of the proposed quadratic expression, will be obtained, namely,

$$x + \frac{1}{2}p + \sqrt{\left(\frac{1}{4}p^2 - q\right)} \dots\dots\dots\dots (2)$$

$$x + \frac{1}{2}p - \sqrt{\left(\frac{1}{4}p^2 - q\right)} \dots\dots\dots\dots (3).$$

These, therefore, are the two factors of the first degree, of which the product is the proposed expression of the second degree; so that each of these equated to zero must furnish the two roots of the quadratic equation; that is, the roots are

$$x = -\frac{1}{2}p \pm \sqrt{\left(\frac{1}{4}p^2 - q\right)}.$$

And this seems to me to be as easy and as elementary a process, for arriving at the formula of solution for a quadratic equation, as the method in common use.

If x^2 have a coefficient other than unity, the steps are much the same; thus, if

$$ax^2 + bx + c = 0;$$

$$\therefore\ 4a^2x^2 + 4abx + 4ac = 0,$$

$$\text{or } 4a^2x^2 - (-4abx - 4ac) = 0,$$

of which the unmodified factors of the first member are

$$2ax \pm \surd(-4abx - 4ac)\,;$$

and these, by the principle above, may be replaced by

$$2ax + b \pm \surd(b^2 - 4ac),$$

the factors of solution, as before.

(8.) I have introduced these simple applications of the general formulæ (I), (II), chiefly for the purpose of preparing the way for a remark of some analytical interest. If the reader turn to the first factor in either of these formulæ, I think he will be disposed to affirm, from an examination of it, that it is impossible for any value of x, which causes X to vanish, to cause *also* the entire factor to vanish; inasmuch as he will perceive that, for $X = 0$, the factor referred to becomes

$$F + \surd F^2, \qquad \text{or } F + f + \surd(F + f)^2,$$

that is,

$$2F, \qquad \text{or } 2(F + f),$$

which is zero only when F or $F + f$ is zero, values of F and $F + f$, which we here suppose to be excluded.

But, from what is done above, we see that this apparent impossibility has been actually effected; for the same value of x, namely,

$$x = -\frac{1}{2}p - \surd\left(\frac{1}{4}p^2 - q\right) \ldots\ldots\ldots\ldots (4),$$

that causes

$$X = x^2 + px + q$$

to vanish, causes also the entire factor (2) to vanish.

For the purpose of clearer illustration, suppose we write this factor (2) in exact accordance with the general form in (II), without any contraction; thus:

$$x + \frac{1}{2}p + \sqrt{x^2 + px + \frac{1}{4}p^2 - (x^2 + px + q)} \ldots (5),$$

or, which is, of course, the same thing,

$$x + \frac{1}{2}p + \sqrt{\left(x + \frac{1}{2}p\right)^2 - (x^2 + px + q)} \ldots\ldots\ldots (6).$$

Now, it is plain, that the value of x, in (4), substituted in x^2+px+q, reduces that expression to zero; and that, suppressing this zero, the entire factor (6) can become zero only for $x+\frac{1}{2}p=0$, or $x=-\frac{1}{2}p$; and therefore cannot become zero, like the factor (2) for the value of x, for which x^2+px+q has vanished, namely, the value (4). And yet (6) and (2), or which is the same thing, (5) and (2), are virtually identical.

The unavoidable conclusion is this, namely: If we substitute in (6) the value of x, exhibited in (4), *before* the subtraction indicated by the minus is executed, we get a certain result,

$$-\sqrt{\frac{1}{4}p^2-q}+\sqrt{\left(-\sqrt{\frac{1}{4}p^2-q}\right)^2}=-2\sqrt{\frac{1}{4}p^2-q};$$

but, if we make the substitution *after* the subtraction is performed, we get a different result, namely, *zero*.

This is a truth little likely to have been anticipated; but it is only one among the many apparent paradoxes of analysis which seem to force themselves upon the attention of the investigator, to teach him the wholesome lesson of caution and discrimination. The expression under the radical sign in (6) is not a perfect square; and therefore, in taking the square root, the *plus* before the radical is not overruled by any law of generation controlling that expression. Before anything is substituted for x, we perceive that this is a *general* truth respecting it; and that, therefore, from the prefixed sign, the root, when extracted, is to be taken *positively*. If, by any management or contrivance, we force, in a particular case, a violation of a *general* law, I need scarcely say that our result will be inadmissible.

(9.) Trifling as the matter here dwelt upon may appear to some, it is nevertheless one among the many delicate points of analysis in which the cultivators of pure science take a peculiar interest. The examination of such points generally calls forth an increased degree of that careful scrutiny and wary circumspection, which analytical investigations so often demand. He who desires to cultivate his reasoning powers, his penetration, his sagacity,

should study geometry; but if he wish to bring all these into still more vigorous exercise, he must apply himself to analysis.*

(10.) Returning from this digression, I shall now proceed with the general investigation:

Let

$$x^{2n} + ax^{2n-1} + bx^{2n-2} + cx^{2n-3} + dx^{2n-4} + \ldots + k = 0 \ldots (1)$$

be any equation of an even degree, into which every equation of an odd degree may be converted by multiplying all its terms by x; the following expressions obviously form a pair of conjugate factors of the left-hand member of the equation, namely,

$$x^n \pm \sqrt{\{-ax^{2n-1} - bx^{2n-2} - cx^{2n-3} - dx^{2n-4} - \ldots - k\}} \ldots (2),$$

and, in virtue of the principle (II), these expressions may be successively changed into the following forms:

$$x^n + \frac{1}{2}ax^{n-1}$$
$$\pm \sqrt{\left\{\left(\frac{1}{4}a^2 - b\right)x^{2n-2} - cx^{2n-3} - dx^{2n-4} - \ldots - k\right\}} \ldots (3),$$

$$x^n + \frac{1}{2}ax^{n-1} - \frac{1}{2}\left(\frac{1}{4}a^2 - b\right)x^{n-2}$$
$$\pm \sqrt{\left\{-\frac{1}{2}a\left(\frac{1}{4}a^2 - b + c\right)x^{2n-3} + \frac{1}{4}\left[\frac{1}{4}\left(\frac{1}{4}a^2 - b\right)^2 - d\right]x^{2n-4} - \ldots - k\right\}} \ldots (4),$$

* This is in direct opposition to the opinions of the distinguished writer, already alluded to; and I think many, besides myself, will regret that so eminent a person should have published such views as those contained in his recent work, 'On a Liberal Education.' The statements pervading that book, and the arguments by which they are defended, are, I think, anything but satisfactory and conclusive. Dr. WHEWELL says that "analytical operations in mathematics do not discipline the reason." Few persons are likely to concur with him in this assertion. But if the reason only is to be disciplined, let the disciple read Euclid; if this prove insufficient, let him read it again. But to dictate that the Principia should be studied as NEWTON wrote it, is to put the dial of science "fifteen degrees backward." The results of NEWTON are imperishable truths; the methods by which he reached them are, for the most part, obsolete.

and so on; the principle of these successive changes being, that we are to take care always to introduce into the rational part, common to both conjugate factors, a quantity, such that twice the product of it, by the leading term, may be equal to the leading term under the radical, and of opposite sign.

It is plain that, by continuing this process, we shall at length arrive at a pair of conjugate factors, such that the expression under the radical will be of the degree $n - 1$; so that when the proposed equation is of the fourth degree, the expression under the radical, in the corresponding conjugate factors, may be reduced to the first degree; when the proposed equation is of the sixth degree, the expression under the radical may be reduced to the second degree, and so on.

(11.) Let now the first member of the proposed equation be represented by X, and the rational part, common to any pair of its conjugate factors indifferently, by X_1; put also X^1 for the expression under the radical sign. Then we shall have

$$X = (X_1 + \sqrt{X^1})(X_1 - \sqrt{X^1}) = 0$$

and since each root of the equation $X = 0$ must satisfy one or other of the conditions

$$X_1 + \sqrt{X^1} = 0$$
$$X_1 - \sqrt{X^1} = 0,$$

we deduce the following obvious but important principle, namely, that, in seeking the places of the real roots of $X = 0$, all those values of x, which cause X^1 to become negative, must be rejected; and which principle may be expressed by the following

THEOREM.

(12.) In seeking the places of the real roots of $X = 0$, all those intervals from $x = a$ to $x = b$, throughout which X^1 continues negative, must be rejected; and those only retained for further examination, throughout which X^1 continues positive.

We thus see the importance of examining the auxiliary and comparatively simple function X^1, before proceeding to discuss

the doubtful intervals which, in equations of a high degree, usually occur in the partial analysis furnished by BUDAN's functions; that is, by the several limiting or derived palynomials deduced from X. The method I am explaining is not wholly independent of all reference to these polynomials, or rather to this preliminary process of BUDAN; but, in the use hitherto made of BUDAN's theorem, whether for the purpose of preparing the way for a final and complete analysis by the method of FOURIER, or by that which BUDAN himself devised for the same purpose, a great deal of tedious labour, must in general, be expended upon the successive contraction of intervals within which we, at length, find no roots can possibly exist. Those who have had much experience of this laborious waste of calculation, will readily appreciate the value of any external guidance in a procedure of so much uncertainty; and in search after real roots, within the limits of their existence, as assigned by BUDAN's theorem, will be glad to avail themselves of any indications which may mark with certainty unproductive intervals.

(13.) Let us now examine into the general theory of the auxiliary function X^1, it being however understood that, in this examination, the object is simply to ascertain its character and peculiarities under different circumstances, and not to describe the steps necessary to be taken in completing the analysis of an equation.

Let us imagine, then, that, for any equation $X=0$ that may be proposed, we have BUDAN's partial analysis before us. Those intervals in this analysis, which comprehend only single roots, furnish definite information of the fact, and require no further examination; those which supply indications of pairs of roots, are the *doubtful* intervals; and it is the character of these alone that remains to be ascertained. If, for the values of x throughout any one of these intervals, X^1 continue *negative*, the doubt is removed; and we may at once conclude that the roots indicated are imaginary. It may happen that, throughout all the intervals within which the doubtful indications lie, X^1 continues negative: we may then safely infer that all the pairs of roots indicated are imaginary.

Should we find that X^1 is such that $X^1 = 0$ has all its roots imaginary, then, provided only that the leading sign in X^1 be negative, we may conclude, without any reference at all to BUDAN's intervals, that the equation $X = 0$ has no real roots.

Suppose we were actually to solve the equation $X^1 = 0$; we should then ascertain precisely the limits which separate the positive and negative intervals in the function X^1; and should pronounce those pairs of roots of $X = 0$, indicated by BUDAN's analysis, in any of the latter intervals, to be imaginary.

It may happen that the roots of the equation $X^1 = 0$, or some of them, may be *equal*. In this case, the equation $X = 0$ will offer some interesting particulars. First, suppose the equation to be of the sixth degree, for which, as we have seen (p. 12), $X^1 = 0$ may be reduced to a quadratic. If the roots of this quadratic be equal, then X^1 being a complete square, it follows, that if it be preceded by the minus sign under the radical, the equation $X = 0$ can have none but imaginary roots; except, indeed, the roots of $X^1 = 0$ equally belong to $X = 0$; that is, except X be divisible by X^1; or, which is the same thing, X_1, by $\sqrt{X^1}$. In this case the remaining roots, all of course imaginary, will be given by the biquadratic equation

$$\frac{X}{X^1} = 0 \text{ or } \frac{(X_1)^2}{X^1} + 1 = 0$$

If, however, in the case of equal roots of $X^1 = 0$, the sign of X^1 be plus, so as to furnish no rejective intervals, then the proposed equation becomes decomposable into the two cubic equations

$$X_1 + \sqrt{X^1} = 0$$
$$X_1 - \sqrt{X^1} = 0.$$

It is scarcely necessary to observe that, in all cases, the product of every pair of conjugate factors, into which a function X is decomposed, always reproduces that function in one or other of the forms

$$X = (X_1)^2 + X^1$$
$$X = (X_1)^2 - X^1.$$

If the equation of the sixth degree, which we have been here

discussing, were originally one of the fifth degree, converted into the sixth by multiplying all its terms by x, then X^1, being in this case necessarily of the form

$$X^1 = ax^2 + bx + \beta^2,$$

in which the last term is essentially positive, could not be a square preceded by the minus sign, whatever signs a and b may have: the sign of the square would always be plus, and $\sqrt{X^1}$ would be of the form $px + \beta$, or $px - \beta$; and since β enters into X_1, as well as here, we should have, after suppressing the x in that conjugate factor in which β becomes cancelled, a decomposition of the equation of the fifth degree into two others of the following forms, namely:

$$x^2 + gx + k = 0$$

$$x^3 + gx^2 + mx + 2\beta = 0,$$

in which all the coefficients are known. And it is easy to see in what way these deductions become modified, when the equations under discussion are of the third or fourth degree, instead of those of the fifth or sixth.* But, in reference to equations of the fourth and sixth degrees, there are still some circumstances, suggested by the preceding examination, which seems to deserve special notice. The consideration of these, however, is not to be understood as much advancing the object more immediately aimed at in the present inquiry, but rather as supplying collateral information, which we may as well stop a moment to pick up by the way.

(14.) We have hitherto been reasoning upon the hypothesis of certain definite and distinct conditions as to the composition of the function X^1. When, in actual practice, these are not accurately fulfilled, there may still be an approach to them; and we

* In equations of the fourth degree, the final form of X^1, that is, the form which terminates the process, is only of the first degree; the examination, therefore, of this final form, as to its change of sign, will be very easy. But, for reasons to be hereafter given, it will in general be more satisfactory to stop the operation at the quadratic form of X^1.

may anticipate consequences more or less approximative to those which here follow with unerring precision.

Suppose that, in an equation of the sixth degree—and the observations founded upon this supposition apply equally to an equation of the fourth degree—suppose that the quadratic function X^1, under the radical, and preceded by the minus sign, as above, be not a complete square; but that the equation $X^1 = 0$ has unequal roots, $x = a$, $x = b$, of the same sign.* Then, by the general theorem already given (p. 12), the equation $X = 0$ can have no real roots, except such as may exist between the limits a, b; this is evident, because, from $x = \infty$, down to that one of these limits which is nearest to ∞, X^1, omitting the prefixed minus must continue plus; and from $x = -\infty$, up to the other limit, it must also continue plus; so that, taking the prefixed minus into account, the function under the radical, throughout these ranges, remains minus.

If, upon transforming the equation $X = 0$ for the above-mentioned limits, there be no loss of variation, we, of course, conclude that the roots of that equation are all imaginary; but if a loss of variations takes place, we may then apply the following considerations, bearing in mind that the lost variations must be even in number, because *all* the real roots must lie within the limits a, b.

The final sign of X^1, here supposed to be included in a vinculum, with the minus sign before it, is necessarily *plus*, because the roots of $X^1 = 0$ have, by hypothesis, the same sign; also the final sign in each of the transformations (a) and (b), of the original equation $X = 0$, must be plus, because no real roots of $X = 0$ lie *without* the limits of a and b.

Now it is plain that, by modifying the final term of the proposed equation $X = 0$—which final term, of course, enters, with contrary sign, that of X^1 in each of the conjugate factors—we may render the roots of $X^1 = 0$, thus changed, *equal*; and the foregoing inferences will then apply accurately to the modified

* We need not examine into the case in which the signs of a and b are different, because, in the analysis of an equation, we always discuss the positive and negative intervals separately.

equation $X=0$:—all the roots of it, on each side of the limit $x=p$, p representing one of the equal roots thus brought about, must be imaginary; and although, as we have seen (p. 14), p may be a root of the changed equation, it is impossible that it can be a root of the original equation, *because* of this change.

Whether or not p be a root of this modified equation, may be ascertained with tolerable ease: since, as shown above (p. 14), it cannot be a root of it without also being a root of the corresponding equation $X_1=0$; which, when the proposed is of the sixth degree, is only a cubic; and when the proposed is of the fourth degree, only a quadratic, reducible to a simple equation by dividing by x.

Suppose it turns out that p *is* a root of $X_1=0$; then, since in the original equation $X=0$, the unmodified final term is positive, if this final term have been *increased* to bring about the above-mentioned equality of roots in $X^1=0$, we may infer that a pair of roots in the doubtful interval $[a, b]$ are *real;* for if they were all imaginary, no *increase* of the final term, already positive, could ever render any of them real, much less equal. (See 'Theory of Equations,' page 161.)

The equal roots p would thus be those into which two unequal roots of the original equation $X=0$ have changed, in consequence of the change which has been made in the final term of that equation.

If, however, the final term of X require to be *diminished* to render the roots of $X^1=0$ equal, then, of course, all the roots of $X=0$ must be imaginary.

(15.) It would not be difficult to generalize this inquiry, and extend it to the case of any equation of which a pair of roots is indicated between two values a and b that are roots of $X^1=0$; but, as already observed, the prosecution of the theory in this direction would not much contribute to advance the object of the present discussion. It may not be amiss, however, here to remind the reader that two distinct classes of imaginary roots may exist in the same equation,—the one class having immediate reference to, and depending on, the final term of that equation,

so that a change in this final term alone, without disturbing the other coefficients, will convert two such imaginary roots into a real pair. The other class is independent of this final coefficient, and the imaginaries belonging to that class—though under a modified form—would still enter the equation, however the final term be changed, and even though it be removed altogether.

(16.) FOURIER has dwelt upon these peculiarities at sufficient length to render any further reference to them here superfluous; and I thus briefly allude to them for the purpose of observing that the imaginary roots, spoken of above as indicated between a and b, are only those of the first class here mentioned. Second class imaginaries, in the primitive equation, become first class imaginaries in some of the derived or limiting equations, to which equations the preceding examination may be applied in succession.* And it is pretty obvious, from what has here been said, that although no first class imaginary roots may be indicated between the limits a and b, yet the indications of secondary pairs may lie concealed in that interval,—a fact which it is well to bear in recollection, seeing that, in the present mode of discovering rejective intervals, the final term enters as an essential element of the process. Yet it must not be inferred that the practical advantages of this process disappear, even when the above distinctions are neglected; but only that to obtain, in certain cases, the largest amount of information respecting the texture of an equation, and the character of its roots, they must, in general, be attended to.

It may be noticed further, that in speaking as above of the

* The eminent analyst alluded to above, though fully recognising the marked difference between the two classes of imaginary roots, mentioned in the text, does not distinguish them, as I have thought it convenient to do. It seems desirable that some such distinctive appellations should be adopted; it might, perhaps, be allowable to call the roots of the first class, *primitive* imaginary pairs; and those of the second class, *secondary* imaginary pairs. We should thus sufficiently indicate the fact, that the imaginaries of the first kind discover themselves, in a direct manner, from an examination of the primitive equation; and those of the second kind, only in an indirect manner, from examining, in like manner, the derived or secondary equations.

substitution of the limits a, b, the roots of the quadratic equation, $X^1 = 0$, I do not necessarily mean the employment accurately of these numbers: any other pair of numbers which embrace these will do just as well, provided they do not leave between them an interval so great as to inclose indications of other roots. But, as stated at the outset, I am not now undertaking to point out the most desirable method of proceeding when, in actual practice, we meet with a critical case requiring a more than ordinarily minute examination. It is not my object here to give directions for performing, in a new manner, what may be as readily, or, perhaps, even more readily, effected by existing methods. My purpose is to do away, as much as possible, with the intolerable labour of pursuing roots through regions where they cannot possibly be found, by furnishing decisive indications of barren intervals. The rejection of these is the saving of so much profitless work; the intervals that remain being all that require any examination; the roots in these, as just remarked, may, in peculiar circumstances, be slow in unfolding their character; but I have elsewhere given the results of much careful and continuous consideration of these delicate cases; and my settled conviction is that, in general, the best way of proceeding is to treat the doubtful roots, in such cases, as if they were known to be real; that is, to enter upon their actual development: then, whether they are imaginary or not, is a circumstance that will unambiguously declare itself by indications, occurring sooner or later, which I have sufficiently explained and exemplified.*

I have but one observation more to make:—The limits a, b, above, have been considered to have the same sign, because in Budan's partial analysis—and, indeed, in every analogous method—the several intervals lie each of them wholly in one region; that is, wholly in the positive region, or wholly in the negative region. It appears, from what has here been shown, that when these limits a, b, are roots of the quadratic equation $X^1 = 0$, and that the minus sign is prefixed to X^1, all the roots of the equation $X = 0$, of which the indications lie *without* those limits, are

* See the 'Theory and Solution of Equations of the Higher Orders,' p. 298; and the 'Researches respecting the Imaginary Roots of Numerical Equations.'

imaginary; if two only are indicated *within* the same limits, we have seen how their character may be ascertained; should they prove to be imaginary, we may conclude, not only that they are imaginary, but that they belong to the first class of imaginaries.

If a very trifling modification of the final term of X would convert these into equal real roots, such equal roots might be received as approximate solutions (Theory of Equations, p. 310).

(16.) The foregoing details respecting the minute analysis of a doubtful interval, as well as the more general considerations which precede them, I have entered upon, mainly on account of their theoretical interest, as throwing some light on the constitution and peculiarities of what I have called the conjugate factors of an equation, and as illustrative of the nature of the information, respecting the composition of an equation, which those factors are capable of supplying.

But there are some deductions, arising out of these general considerations, which claim more than a mere passing notice, as they are practically useful as well as theoretically interesting. I have deferred the formal enunciation of them in order, first, to establish another method of obtaining the functions X_1, X^1; as this other method will suggest what some may consider a more convenient form of expression for the theorems I have in view.

If we suppose the general function X^1, under the radical, to become spontaneously zero, then, as each conjugate factor reduces in that case simply to X_1; we infer that X_1 is the complete square root of X; in other words, that X is a complete square. It follows, therefore, in all cases, that X^1 is the quantity to be added to X to make this latter polynomial a complete square; so that $-X^1$ is nothing else than what remains when the operation of extracting the square root of X is stopped; that is, $-X^1$ is the correction which must be applied *subtractively* to X, in order to render that function a complete square. And thus the improvements in the analysis of an equation, which will presently be exemplified, would have been thought of before, if it had only occurred to any one to extract the square root of the first member

of that equation, and then to discuss the character of the remainder; and, as a collateral circumstance, a simple and compact method of extracting the square root of a polynomial—much more convenient than the straggling process in common use—would probably, at the same time, have suggested itself.

(17.) It will, perhaps, be an excusable digression if I offer here an illustration or two of this improved arrangement; observing that, as the remainder due to each step of the operation is to be exhibited with the proper signs, as in the common method, the terms which contribute to the formation of X^1, as already explained, are now to be *subtracted*, not *added*, as in the foregoing pages.

Let the polynomial proposed for extraction be

$$x^6 + 8x^5 - 4x^4 + 2x^3 - 6x^2 - 20x + 17,$$

then the successive approximations to the root, accompanied with the corresponding remainders, or portions of the polynomial still unexhausted, may be arranged in the following manner, upon the principle explained at the commencement of the present discussion (p. 7), the observation above, respecting the formation of the remainder, being attended to.

Root.	*Remainders.*
x^3+4x^2	$-20x^4+2x^3-6x^2-20x+17$
x^3+4x^2-10x	$82x^3-106x^2-20x+17$
$x^3+4x^2-10x+41$	$-434x^2+800x-41^2+17$

And, if requisite, this approximation may be continued on for negative powers of x.

Should the leading term of the proposed polynomial require a factor to make it a square, we may introduce this factor into each of the terms; and if, after the extraction, continued as far as may be thought necessary, we divide the root by the root of the factor, and the remainder by the factor itself, the results will be correct.

As another example, let it be required to extract the square root of

$$x^2 + x + 1.$$

Root.	*Remainder.*
$x + \frac{1}{2}$	$\frac{3}{4}$

so that, taking the remainder, with a changed sign, the conjugate factors of the proposed expression are

$$\left(x+\frac{1}{2}\right)+\sqrt{-\frac{3}{4}}, \text{ and } \left(x+\frac{1}{2}\right)-\sqrt{-\frac{3}{4}}.$$

Lastly, let the polynomial be

$$x^4 + ax^3 + bx^2 + cx + d.$$

Root.	*Remainder.*
$x^2+\frac{1}{2}\,ax$	$\left(b-\frac{1}{4}a^2\right)x^2+cx+d,$
$x^2+\frac{1}{2}ax+\frac{1}{2}\left(b-\frac{1}{4}a^2\right),$	$\left\{-\frac{1}{2}a\left(b-\frac{1}{4}a^2\right)+c\right\}x-\frac{1}{4}\left(b-\frac{1}{4}a^2\right)^2+d,$

from which the conjugate factors of the polynomial may be obtained as before.

It thus appears that the process for the extraction of the square root of a polynomial, whether conducted as above, or by the common method, not only furnishes us with the exact root, when the proposed expression is a complete square, but, by analysing that expression, when not a complete square, into conjugate factors, it enables us to ascertain whether or not it is composed of the sum or difference of two squares; a fact well worthy of notice, although I believe hitherto unobserved. The polynomial, in the last example, is composed of the sum or difference of two squares if

$$4\left(b-\frac{1}{4}a^2\right)d=c^2,$$

the *sum*, if the two factors in the first member of this equation are positive, and the *difference* if they are negative.

(18.) It must not, however, be supposed that when the polynomial is thus constituted, the decomposition here adverted to will invariably be effected by the particular conjugate factors to which our attention has been exclusively confined in the present inquiry. As noticed at the outset, the pairs hitherto dwelt upon —and which are those we shall almost entirely restrict ourselves to throughout—are not the only conjugate factors that may be employed; we may, indeed, decompose a polynomial into as many distinct pairs of conjugate factors as we please; as is evident from the arbitrary character of F and f in the general form at page 7. When, therefore, a polynomial is suspected to be composed of the sum or difference of two squares, or when our object is to put the matter to the test, we may find room, in certain cases, for the exercise of some ingenuity in the discovery of the proper component factors. An easy example will, perhaps, sufficiently illustrate this. Suppose the polynomial were

$$x^4 - 4x^3 + 7x^2 - 4x + 1,$$

then, proceeding as in the foregoing instances, we have the following results, namely,

Root.	*Remainder.*
$x^2 - 2x$	$3x^2 - 4x + 1$
$x^2 - 2x + \frac{3}{2}$	$2x - 1\frac{1}{4}$

The last remainder shows that the expression cannot be a complete square; but we cannot affirm that it is neither the sum nor the difference of two squares from the form of the preceding remainder. To come to a decisive conclusion on this point, we must first ascertain whether *any* quantity, introduced into the partial root $x^2 - 2x$, can render the remainder a square. And it is easy to see that 1 is such a quantity; so that, varying the second step, the results may be written thus:

Root.	*Remainder.*
$x^2 - 2x$	$3x^2 - 4x + 1$
$x^2 - 2x + 1$	x^2

from which we infer that the proposed polynomial is the same as

$$(x^2 - 2x + 1)^2 + x^2.$$

If the polynomial, here discussed, were the first member of a biquadratic equation, that equation might, therefore, be replaced by the two quadratics

$$x^2 - (2 - \sqrt{-1})\,x + 1 = 0,$$

and

$$x^2 - (2 + \sqrt{-1})\,x + 1 = 0.$$

And similarly in other like instances.

(19.) The theorems to which I have prospectively alluded above, in addition to what has already been shown, will require the proof of the following proposition, which, in itself, is not without independent interest.

Every equation of the form

$$(\mathrm{F}x)^2 - fx = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (\mathrm{A})$$

must necessarily have two real roots, provided Fx be of an odd degree, that (Fx)^2 be of higher degree than fx, and that the roots of $fx = 0$ are all imaginary.

For, since (Fx)^2 is of higher degree than fx, a real value so great may be given to x, such as to render the left hand member of (A) *positive.* But, as Fx is of an odd degree, a real value may be given to x, such as to render (Fx)^2 zero; in which case, seeing that fx is invariably positive, the left-hand member will become negative Hence the equation must have one real root; and, since it is of an even degree, it must necessarily have another; and it is plain that we may amplify the proposition by the following addition to it, namely:

And if Fx be of an even degree, the other conditions remaining, the equation (A) will still have a pair of real roots, provided the roots of F$x = 0$ be not all imaginary.

(20.) This proposition, taken in conjunction with what has previously been proved, warrants the inference of the following theorems, namely:

THEOREM I.

If, in an equation of the sixth degree, we arrange the terms all on one side, and extract the square root of the polynomial, then, if the quadratic remainder have its leading sign positive, and its roots imaginary, the proposed equation cannot have a real root.

But if the leading sign be negative, the roots of the quadratic being still imaginary, then the proposed equation must have one pair of real roots at least. And, generally, if the equation be of any even degree, of which the square root is of an odd degree, and that any remainder at which we stop have all its roots imaginary, then, if the leading sign of this remainder be positive, the equation cannot have any real roots; but if the leading sign be negative, it must have one pair of real roots at least.

THEOREM II.

If, in any equation of an even degree, of which the square root of the polynomial forming the first member is also of an even degree, any remainder at which we stop have only imaginary roots; then, if the leading sign of this remainder be positive, the proposed equation cannot have any real roots. But if the leading sign be negative, then the equation must have one pair of real roots at least, provided the roots of the polynomial, which is the result of the extraction, be not all imaginary.

It is further obvious that the two real roots, shown in these theorems to exist under the prescribed conditions, must be unequal roots; for the left-hand member of (A) could not furnish a change of sign for two substitutions for x, of numbers between which equal roots only were comprehended; and, as before noticed, the presence of one root in an equation of an even degree, as indicated by such change of sign, necessarily implies the existence of another root of different value.

It must be noted, too, that in these theorems, the *remainders*

spoken of are those that present themselves in the operation for extracting the square root, whether that operation be conducted in the usual way, or as above exhibited. The function X^1, in the preceding pages, as also in those which are to follow, is this remainder with changed signs; and it is of importance to keep this in remembrance when any reference to these theorems may be made hereafter.

(21.) It is the consideration of this function X^1, as we have already seen, that guides us to the rejective intervals between the extreme limits of the roots. When such rejective intervals exist in conjunction with others not rejective, then at each boundary, between the two sorts of intervals, X^1, of course, vanishes; and the corresponding value of x makes X *plus*, unless, indeed, this same value of x cause X_1 to vanish also, in which case that value will be one of a pair of equal roots of the equation $X=0$. Hence, whether this happen or not, if $X_1=0$ only have a root in a non-rejective interval, for such root, as X_1 vanishes, X must be *minus*. And thus we obtain a third theorem, namely:

THEOREM III.

If $X_1=0$ have a root in any non-rejective interval, the proposed equation, $X=0$, must have real roots (a *pair* of real roots at least, X being of even degree); one will lie between the root of $X_1=0$ and $+\infty$, and another between the same root and $-\infty$. If there be any rejective intervals at all, a real root will be situated between the root of $X_1=0$, here spoken of, and the boundary of that rejective interval which is nearest to it, on either side; so that if a boundary exist on each side, the situation of two real roots within defined limits will be ascertained, X being, of course, understood to be of even degree.

I venture to regard the theorems here announced as of some importance in the analysis of equations. They are, moreover, a little remarkable; since, although we have several criteria for

detecting the presence of imaginary roots, yet, beyond the quadratic, if we set aside the theorem of STURM, I believe there exists no rule for enabling us to affirm the presence of *real* roots, in any equation of an even degree, with the last term positive, unless, indeed, the utterly impracticable tests first proposed by WARING, and afterwards re-discovered by LAGRANGE, can be regarded as an exception.

(22.) There is frequently a strong temptation, in the investigation of such criteria, to assume a converse proposition from a direct one. I think MACLAURIN must have yielded to some such temptation when he undertook the demonstration of NEWTON'S rules for the discovery of imaginary roots; for he commences the inquiry evidently under the impression that these rules apply both directly and conversely; an impression, however, which meets with no confirmation as he goes on, and which he closes his paper without undertaking to justify.

Most persons engaged in these reseaches, I dare say, have had sedulously to guard against similar temptations. I acknowledge that, for myself, even in the course of the present discussion, views and considerations have presented themselves with such plausible allurement, and supported by so many examples devised to test their validity, that I have found some resolution to be necessary to escape the snare, and to keep myself honestly within the pale of rigorous demonstration. And I hope it will be found, accordingly, that I have here advanced nothing respecting the real roots of an equation, merely from the non-detection of imaginary roots. All the positive information, which I have felt justified in deducing, about the real roots of an equation, is comprised in the theorems here given, and in a few other general inferences that will be found towards the close of the essay. But I entertain the expectation that additional theorems of interest may be hereafter derived from a further examination of the conjugate factors into which it is here proposed to divide the first member of an equation.

(23.) I shall now proceed to practical applications of the theory

developed in what has preceded; and, in order to secure the more ready means of instituting a comparison between existing methods and that which this theory suggests, I shall, in general, take examples which have already been discussed by BUDAN, FOURIER, STURM, or other writers, who may have employed their processes. As I shall quote the authors from whom these examples are extracted, I do not conceive that it will be at all necessary that I should encumber these pages by copying from those authors the initial steps of the analysis of an equation by the theorem of BUDAN. It will, indeed, be seen, in the subsequent operations, that we may frequently dispense with the theorem altogether; for, in the present method, it will usually be best to ascertain the rejective intervals, by an examination of our function X^1 before any portion of BUDAN's analysis is undertaken, as it is of no use to search out the particular intervals furnished by that analysis, within which roots, already known to be imaginary, are indicated. Neither shall I, in the course of the analyses to be given, pay much attention to the criteria already in existence for testing the character of the roots of an equation by an examination of the coefficients, as I wish the method here proposed to be judged of, independently of such extraneous aid. Of the criteria adverted to, those given, without demonstration by NEWTON, in his 'Arithmetica Universalis', and which have been glanced at above, are, perhaps, the most important; but, no proof of their truth having been given till very recently, they have been altogether neglected by modern analysts. Their demonstration, as already noticed, was undertaken by MACLAURIN; and MONTUCLA seems to think with success;* though a careful examination of their reasoning will show that all that MACLAURIN and CAMPBELL did in this way is insufficient to enable us to detect, with certainty, more than *two* imaginary roots of an equation, even under the most favorable circumstances. A complete demonstration of NEWTON's rule, was, I believe, for the first time, given in my 'Researches respecting the Imaginary roots of Numerical Equations,' published in 1844. Other rules, analogous to this of NEWTON, will be given towards the close of the present essay.

* Histoire des Mathématiques; tome iii, p. 31.

EXAMPLES.

1. Fourier, 'Analyse des Equations,' p. 137,

$$x^4 - 4x^3 - 3x + 23 = 0.$$

The proper conjugate factors of the first member of this equation are

$$x^2 - 2x \pm \surd(4x^2 + 3x - 23)$$
$$= x^2 - 2x - 2 \pm \surd(7x - 19),$$
$$\therefore \mathrm{X}^1 = 7x - 19.$$

As, for all negative values of x, X^1 is negative, we infer that the equation cannot have any negative roots. And as X^1 continues negative for all positive values of x, up to 2 inclusive, it further follows that, in substituting numbers below 10, the superior limit of the roots in Budan's (or Fourier's) functions, we may commence with 2. The substitution of this number furnishes but two variations of signs. We conclude, therefore, that the equation has two imaginary roots. The substitution of 3 causes one of these variations to disappear, so that the equation has two real roots, of which one lies between 2 and 3, and the other between 3 and 10. (See Fourier, pp. 107 and 138.)

Fourier arrives at these results only after a laborious process, involving an operation for the common measure of X, and its first derived function.

It will be observed that the preceding expression for X^1 shows that the root, afterwards found to lie in the interval [2, 3], lies in the still narrower interval [$2\frac{5}{7}$, 3]; a circumstance not unworthy of notice when actual development is the object.

2. Fourier, page 143,

$$x^4 - x^3 + 4x^2 + x - 4 = 0$$

$$x^2 - \frac{1}{2}x \pm \surd\left(-\frac{15}{4}x^2 - x + 4\right),$$

$$\therefore \mathrm{X}^1 = -\left(\frac{15}{4}x^2 + x - 4\right).$$

It is obvious, without proceeding further, that since X^1 is negative for $x = 1$, and for all higher values, the positive roots of the equation must all lie between 0 and 1.

Applying these limits in BUDAN's functions, one variation is lost, and two variations still remain; and since, as just shown, these cannot be lost by the passage of *real* roots, we infer that there is a pair of imaginary roots in the positive region; and, consequently, that there is a single real root in the negative region.

(3.) LOCKHART. 'Resolution of Equations,' p. 26,

$$12x^3 - 120x^2 + 326x - 127 = 0,$$

or,

$$36x^4 - 360x^3 + 978x^2 - 381x = 0$$

$$6x^2 - 30x \pm \sqrt{(-78x^2 + 381x)}$$

$$6x^2 - 30x + \frac{78}{12} \pm \sqrt{-\left(9x - \frac{13^2}{4}\right)},$$

$$\therefore\ X^1 = -9x + \frac{13^2}{4}.$$

Hence X^1 is negative from $x = \frac{13^2}{4} \div 9$, onwards; that is, from $4{\cdot}7$ to ∞. But BUDAN's functions give two variations for $4{\cdot}7$; hence the roots in the positive region are imaginary.

This example, although only of the third degree, is one of considerable difficulty. According to Mr. LOCKHART, the complete analysis of it, by BUDAN's method, is all but impracticable;* and it is also exceedingly troublesome by the process of FOURIER, as I have elsewhere shown.†

(4.) Let it be proposed to determine the criteria of imaginary roots in a general equation of the fourth degree, with its last term positive,

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

* Lockhart's 'Resolution of Equations;' Oxford, 1837, p. 28.

† Theory and Solution of Equations of the Higher Orders; p. 244.

$$x^2 + \frac{1}{2} ax \pm \sqrt{-\left(b - \frac{1}{4} a^2\right) x^2 - cx - d},$$

$$\therefore\ X^1 = -\left\{\left(b - \frac{1}{4} a^2\right) x^2 + cx + d\right\}.$$

Consequently, if

$$4 \left(b - \frac{1}{4} a^2\right) d > c^2,$$

the roots are all imaginary.

So simple a criterion as this has not, I believe, been given before.

If the second term ax^3, be absent from the equation, then the criterion becomes

$$4bd > c^2,$$

in which case, we have a confirmation of NEWTON'S rule; for, since d is necessarily positive, b must be positive likewise; so that, as we know, from the criterion of DE GUA, that two imaginary roots are indicated by the absence of the second term, those on each side having like signs, and as two are also indicated by the three terminating coefficients, and none by the middle ones, NEWTON'S conditions are fulfilled; and, as here shown, the roots are, in consequence, all imaginary.

There never existed any doubt as to NEWTON'S rule indicating a *pair* of imaginary roots, when these conditions are fulfilled; but, till recently, as before noticed, it was matter of question whether we could ever safely infer the existence of *more* than one pair.

The roots of the preceding equation are still imaginary, though the condition should be

$$4 \left(b - \frac{1}{4} a^2\right) d = c^2,$$

unless the roots of $X^1 = 0$ also satisfy the equation

$$x^2 + \frac{1}{2} ax = 0,$$

in which case, the proposed equation would have a pair of roots, each equal to $-\frac{1}{2}a$; the other two roots must be imaginary (page 14).

It may not be superfluous to remark here, in reference to equations of the fourth degree, that all the roots in the positive region will be imaginary, if the roots of $x^2 + ax + b = 0$ are imaginary, and if, moreover, c and d be *positive;* and those in the negative region will be imaginary if, the other condition remaining, c be negative, and d positive; for, under these restrictions, all positive substitutions in the one case, and all negative substitutions in the other, give positive results.

(5.) Midy. 'Du Théorême, de M. Sturm,' p. 35,

$$x^6 + x^5 - x^4 - x^3 + x^2 - x + 1 = 0,$$

$$x^3 + \frac{1}{2}x^2 \pm \sqrt{\frac{5}{4}x^4 + x^3 - x^2 + x - 1}$$

$$x^3 + \frac{1}{2}x^2 - \frac{5}{8}x \pm \sqrt{\frac{3}{8}x^3 + \left(\frac{5^2}{8^2} - 1\right)x^2 + x - 1}$$

$$x^3 + \frac{1}{2}x^2 - \frac{5}{8}x - \frac{3}{16} \pm$$

$$\sqrt{\left(\frac{5^2}{8^2} - 1 - \frac{3}{16}\right)x^2 + \left(1 + \frac{15}{64}\right)x - 1 + \left(\frac{3}{16}\right)^2}$$

$$\therefore\ X^1 = -\left(\frac{51}{64}x^2 - \frac{79}{64}x + \frac{247}{256}\right)$$

or,

$$-(204x^2 - 316x + 247).$$

As this is preceded by the minus sign, and the roots of it are imaginary, we conclude (page 14) that all the roots of the proposed equation are imaginary.

It is in equations such as this, where all, or a considerable proportion, of the roots turn out to be imaginary, that the method here explained discloses, in the most obvious manner, its peculiar

advantages ; and it is precisely in such equations that extra assistance was more especially needed. I have, elsewhere, analysed this equation, both by the theorem of STURM and that of FOURIER, but not without a large amount of very tedious labour, notwithstanding that every expedient to abridge the processes was resorted to.* The work above did not occupy more minutes than these methods required hours; for, in such an extent of calculation as is exhibited in the analysis alluded to, frequent and careful revision of the steps of the work become necessary to ensure accuracy. BUDAN'S method, also applied to this equation, in the same place, happens to be shorter, but is much inferior to that above.

(25.) As it is not my wish that the present mode of treating an equation submitted to analysis should be overrated, I have not resorted to the stratagem of framing examples peculiarly fitted to its powers, but have taken them at random from other works; and I take this occasion again to state, that the value of the method mainly consists in its readily making known to us the rejective intervals in which it would be in vain to seek for real roots. This is always an advantage; but when there happens to be no imaginary roots in the equation proposed, the advantage may then be considered as reduced to its minimum, although it does not wholly disappear. And nearly the same may be said, when, as will sometimes happen, imaginary roots exist in non-rejective intervals. But in no case whatever can the work be entirely fruitless, as the theorems at page 25, and what precedes them, sufficiently show; and as will be still further seen, in a scholium appended to these examples.

As already observed, the pair of roots which are indicated in a non-rejective interval, if submitted to the process of actual development, unfold their character in a perfectly unambiguous manner.

(26.) The following example is selected for the purpose of exhibiting the method in its least favorable aspect.

* See the 'Theory and Solution of the Higher Equations,' p. 234.

(6.) STURM. 'Third Example,'*

$$x^3 + 11x^2 - 102x + 181 = 0,$$

or,

$$x^4 + 11x^3 - 102x^2 + 181x = 0,$$

$$x^2 + \frac{11}{2}x \pm \sqrt{\left(\frac{529}{4}x^2 - 181x\right)};$$

$$\therefore\ X^1 = 529x^2 - 724x.$$

Here we have but a very small rejective interval in the positive region, namely, the interval $\left[0, \frac{724}{529}\right]$; and no rejective interval in the negative region.

In BUDAN's functions, therefore, we commence our substitutions, in the positive region, with $x = 2$; and we find two variations to disappear between 3 and 4. We may, therefore, apply the process of development to the roots thus indicated in a non-rejective interval, and shall find them to turn out real, and to be

$$3{\cdot}213 \ldots\ldots \text{ and } 3{\cdot}229 \ldots\ldots$$

as STURM has determined, but not without much more trouble. As no other root can exist in the positive region, we conclude that the remaining root is negative; and since $x = -\frac{11}{2}$ is a root of

$$X_1 = x^2 + \frac{11}{2}x = 0,$$

we know, from Theorem III, page 26, that this negative root must be greater than $-\frac{11}{2}$. It is found to lie between -10 and -20.

(27.) And here it may be proper to remind the reader that, in dealing with equations of an odd degree, we always, as above,

* See MR. SPILLER's translation: 'The Solution of Numerical Equations, by C. STURM;' p. 30.

introduce an extraneous root, namely, the root $x = 0$; and that the theorem, just referred to, applies to the equation, in its full extent, only after this modification; so that when a root is shown by that theorem to exist between certain assignable limits, we must take notice whether this be only the extraneous root $x=0$, temporarily engaged, or a root really belonging to the original equation. If this point be not first settled, we cannot deduce any conclusion as to the roots of the equation proposed.

(7.) FOURIER, page 145,

$$x^5 + x^4 + x^3 - 2x^2 + 2x - 1 = 0,$$

or,

$$x^6 + x^5 + x^4 - 2x^3 + 2x^2 - x = 0$$

$$x^3 + \frac{1}{2}x^2 \pm \sqrt{\left(-\frac{3}{4}x^4 + 2x^3 - 2x^2 + x\right)}$$

$$x^3 + \frac{1}{2}x^2 + \frac{3}{8}x \pm \sqrt{\left(\frac{19}{8}x^3 - \frac{119}{64}x^2 + x\right)};$$

$$\therefore\ X^1 = (152x^2 - 119x + 64)\,x.$$

As the roots of the quadratic in the vinculum, are imaginary, every substitution for x, made in it, must give a positive sign. Consequently, on account of the other factor x, the substitution of negative numbers must always give a negative sign; hence there are no real roots in the negative region; so that the roots must all be sought in the positive region.

In BUDAN's functions, three variations only are given for $x=0$; or, which is the same thing, without referring to BUDAN's functions at all, three variations only occur in the original equation. Consequently, the equation must have two imaginary roots at least; and these are indicated in the negative region. Indications of the remaining three roots occur in the non-rejective interval $[0, \infty]$. (See page 46, in reference to this.)

As stated at page 18, it will be desirable to reject secondary imaginary roots if there be any; so that, taking the derived equation, for the purpose of ascertaining this, we have

$$5x^4 + 4x^3 + 3x^2 - 4x + 2 = 0$$

or,

$$25x^4 + 20x^3 + 15x^2 - 20x + 10 = 0$$

$$5x^2 + 2x \pm \surd(-11x^2 + 20x - 10)$$

$$\therefore X^1 = -(11^2 - 20x + 10),$$

the roots of which are imaginary; so that the proposed equation has but one real root, and that is positive; as, indeed, we otherwise know from the sign of the final term. Had the interval here, instead of being wholly rejective from $-\infty$ to $+\infty$, been rejective only from 0 to 1, the same conclusion would have followed; since, by BUDAN's functions, a pair of roots of the derived polynomial, here examined, are indicated in that interval.

(28.) As already noticed, the preceding examples are all discussed, without any aid from criteria, respecting imaginary roots, as furnished by the coefficients of the original equation. Assistance, spontaneously offered, has thus been disregarded. But the rule of NEWTON, and that of DE GUA, which is comprehended in it, as they often supply valuable information from a mere inspection of the coefficients of an equation, are of considerable use in particular cases, whatever general method of analysis be employed. If NEWTON's rule had been appealed to in the discussion of the preceding equation, we should have discovered the existence of the four imaginary roots at once. I shall avail myself of the information supplied by this rule in the next example.

(8.) FOURIER, page 111,

$$x^7 - 2x^5 - 3x^3 + 4x^2 - 5x + 6 = 0$$

or

$$x^8 - 2x^6 - 3x^4 + 4x^3 - 5x^2 + 6x = 0$$

$$x^4 - x^2 \pm \surd(4x^4 - 4x^3 + 5x^2 - 6x)$$

$$x^4 - x^2 - 2 \pm \surd(-4x^3 + 9x^2 - 6x + 4);$$

$$\therefore X^1 = -(4x^3 - 9x^2 + 6x - 4).$$

By NEWTON'S rule, the original equation has at least four imaginary roots; although we cannot infer from that rule the particular intervals in which indications of the imaginary pairs lie. The cubic equation,

$$4x^3 - 9x^2 + 6x - 4 = 0,$$

by the same rule, has also a pair of imaginary roots; and we see at once, that its real root lies between 1 and 2; so that [1, 2] is an interval partly rejective and partly non-rejective; and, throughout the entire region, beyond 2, X^1 is rejective. It is further obvious that $\sqrt{2}$ is a root of

$$X_1 = x^4 - x^2 - 2 = 0;$$

and, with a view to the application of Theorem III, it remains to be ascertained whether this root, $\sqrt{2}$, is in the rejective or in the non-rejective part of [1, 2].

For $x = \sqrt{2}$, X^1 is $22 - 14\sqrt{2}$, which is obviously plus; therefore the root $\sqrt{2}$, of $X_1 = 0$, lies in a non-rejective interval, the non-rejective part of [1, 2]; and, consequently, by Theorem III, one real root, at least, must lie between 1 and 2.

By substituting these limits in the proposed equation, we find no change of sign in the results; hence there must be *two* real roots in the interval [1, 2]. We therefore conclude that the equation has two positive roots between 1 and 2, one negative root and four imaginary roots; and this without any reference to BUDAN'S functions.

In this example, NEWTON'S criteria have afforded considerable assistance; but, valuable as they are, in certain circumstances, their capabilities must not be overrated. The unfavorable feature in them is, that, although in such circumstances they make known the existence of a pair of imaginary roots, yet they do not indicate the *interval,* but only the *region,* to which they belong; so that, when pairs of roots occur in different intervals of that region, NEWTON'S rule does not enable us to distinguish the interval of imaginary from the interval of real roots. It is remarkable, however, that no mention of this rule is ever made —even in its simplest application—by any of the modern conti-

nental writers on equations; and even English authors, when they do mention it, content themselves with a mere passing notice of it, with the single exception of the venerable MR. LOCKHART, a gentleman who, at the age of eighty-seven, is still prosecuting his scientific researches with the same ardour that animated his early years.*

The following example is from the "Appendix" to that gentleman's work, on the 'Resolution of Equations by means of Inferior and Superior Limits,' published in 1843.

(9.) LOCKHART. 'Appendix,' &c., page 9.

$$x^4 - 7{\cdot}2x^3 + 13{\cdot}44x^2 - 1{\cdot}728x + {\cdot}0576 = 0,$$

$$x^2 - 3{\cdot}6x \pm\sqrt{(-{\cdot}48x^2 + 1{\cdot}728x - {\cdot}0576)},$$

$$x^2 - 3{\cdot}6x + {\cdot}24 \pm \sqrt{0}.$$

The first member of the proposed equation is, therefore, a complete square; so that it has two pair of equal roots, one of each pair being furnished by the quadratic

$$x^2 - 3{\cdot}6x + {\cdot}24 = 0.$$

The above equation has been introduced for the purpose of showing the decided advantage of this method, whenever the polynomial is a complete square; it always makes known to us whether such is the case or not;—a feature peculiar to the form of the conjugate factors more especially dwelt upon in the present essay.

The reader can readily analyse, in the same way, the equation

$$x^4 - 10x^3 + 31x^2 - 30x + 9 = 0,$$

taken from MR. LOCKHART's tract on 'The Nature of the Roots of Numerical Equations,' alluded to in the foot-note. The component quadratics will be found to be

$$x^2 - 5x + 3 = 0$$

$$x^2 - 5x + 3 = 0$$

* I may here mention, as a fact, not without a certain kind of interest, that MR. LOCKHART is at present (May, 1850) engaged in carrying through the press some of his recent investigations on the Resolution of Equations.

Before terminating these examples, I shall apply the method to the following general problem the discussion of which is not without interest:

(29.) To determine the conditions for which the roots of the general equation of the sixth degree must all be imaginary; the last term of the equation being, of course, positive.

(10.) $x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0,$

$$x^3 + \frac{1}{2}ax^2 \pm \sqrt{\left\{\left(\frac{1}{4}a^2 - b\right)x^4 - cx^3 - dx^2 - ex - f\right\}}$$

or putting $b - \frac{1}{4}a^2 = p$, and dividing by p, the expression within the brackets will become

$$x^4 + \frac{c}{p}x + \frac{d}{p}x^2 + \frac{e}{p}x + \frac{f}{p},$$

and this, equated to zero, will have all its roots imaginary, provided the condition established in example 4 have place; that is, provided

$$4\left\{\frac{d}{p} - \frac{1}{4}\left(\frac{c}{p}\right)^2\right\}\frac{f}{p} > \left(\frac{e}{p}\right)^2, \text{ and } p \text{ positive;}$$

or, removing fractions, and restoring p, provided

$$4\left\{\left(b - \frac{1}{4}a^2\right)d - \frac{1}{4}c^2\right\}f > \left(b - \frac{1}{4}a^2\right)e^2 * \quad \ldots.(A).$$

If $a = 0$, then it is plain that X^1 remains the same, whether x^6 be absent from the original equation or not. In fact, if we merely expunge a, the conjugate factors become

$$x^3 \pm \sqrt{\{-bx^4 - cx^3 - dx^2 - ex - f\}},$$

and if x^6 be also suppressed, they may be written

$$0 \pm \sqrt{\{-bx^4 - cx^3 - dx^2 - ex - f\}},$$

the X^1 being the same in both cases.

* It is, of course, to be understood, as in the criteria of Newton, that the second member of each inequality, and, consequently, the first, must be positive.

Hence, for the biquadratic equation

$$bx^4 + cx^3 + dx^2 + ex + f = 0,$$

the condition (A) becomes

$$4\left\{bd - \frac{1}{4}c^2\right\}f > be^2 \ldots\ldots\ldots\ldots \text{(B)},$$

which, if $b = 1$, becomes,

$$4\left\{d - \frac{1}{4}c^2\right\}f > e^2,$$

as already determined.

In like manner, making $b = 0$, $c = 0$, we have for the quadratic equation,

$$dx^2 + ex + f = 0,$$

the condition

$$4df > e^2,$$

the well known criterion of ordinary algebra.

And this is the only criterion of the absence of all real roots, hitherto discovered in the general theory of equations, which holds both directly and conversely; whenever this has place, the quadratic equation, whose coefficients furnish it, has its roots necessarily imaginary; and when its roots are imaginary, this condition necessarily has place; a circumstance which is connected with the fact that, for a quadratic equation, we have a general formula of solution, not in the least controlled by any special condition; but no such formula for the higher orders of the equations. It is on account of this peculiarity that, even in equations of the fourth degree, examined in the foregoing examples, I have usually preferred to stop the operation when X^1 has reached the second degree, rather than to extend it a step further. It is often prudent, however, in equations of the fourth degree, to glance, at least mentally, at the leading term of X^1 in the final step; for if the last term of X happen to be a complete square, X^1 may wholly disappear in that final step; showing that X itself is a complete square, as in example 9; or, when the last term of X is not a square, still X^1 may vanish in part, and

become reduced to px^2 or to a single *number*. In either case, the proposed equation becomes decomposed into a pair of rational quadratics. But, if fractional coefficients enter X_1, when the coefficients of X are integral, neither of these things can happen.

The following equation will illustrate what is here said; and will also furnish an additional exemplification of the principle noticed at page 23.

The equation was first proposed in 1658, by HUDDE; and is quoted by MR. LOCKHART.

(11.) LOCKHART. 'Nature of the Roots,' &c., p. 14.

$$x^4 + 4x^3 - 3x^2 - 8x + 4 = 0$$

$$x^2 + 2x \pm \sqrt{(7x^2 + 8x - 4)}$$

$$x^2 + 2x - 2 \pm \sqrt{3x^2}.$$

Hence the component quadratics are

$$x^2 + (2 + \sqrt{3})\, x - 2 = 0$$

$$x^2 + (2 - \sqrt{3})\, x - 2 = 0.$$

(30.) Since the criterion (B) above is the same, whether x^6 be absent or present, and as it is obvious that the same would hold generally, for the decomposition here considered, we infer that when the second term is absent in any equation of an even degree, we need examine the criterion only in reference to the terms beyond the absent one to the right, taking care, however, in this case, to divide all the terms by the coefficient of that immediately succeeding the absent one, or, at least, not to omit it in the statement of the condition.

Instead of the second term from the beginning, if the second term from the end, namely ex, should be absent, the condition under which all the roots will be imaginary, is equally simple; for, if $e = 0$ in (A), we must have

$$4\left(b - \frac{1}{4}a^2\right) d > c^2,$$

f, of course, being positive; and which is the same condition as we should get if ex and f were both expunged, and the remaining polynomial reduced to one of the fourth degree.

(31.) Many other criteria, enabling us to affirm, with confidence, the entire absence of real roots in an equation, might be investigated. In the above general equation of the sixth degree, we might have proceeded on to the quadratic, instead of stopping at the biquadratic form of X^1; and we might also consider different pairs of the innumerable systems of conjugate factors into which a polynomial may be decomposed. But I do not think that the resulting general forms would be of much practical importance; as the particular forms, in individual cases, may very easily be obtained, as shown in the special examples already given; from which it will be seen how we may reach conditions under which real roots must be absent from a particular region, either the positive or the negative, as well as those under which they must be absent from both regions.

In equations of an odd degree, we may apply such criteria to the limiting equations, after the manner in which we have treated example 7. And, by thus applying the proper limiting or derived equations, we may determine conditions which, when fulfilled, will enable us to affirm that an equation, of the degree $2n$ or $2n + 1$, must have 2, 4, 6, &c., or $2n$ imaginary roots.

Those who are familiar with the writings of MACLAURIN, know that the limiting equations, here alluded to, may be made to assume a variety of forms; so that these criteria will be susceptible of different changes. Into the consideration of these, however, as just remarked, I do not conceive it to be of any importance to enter; and with thus briefly alluding to the general deductions, which any one can readily make for himself, after what has here been done, I shall terminate this part of the inquiry; simply noticing another form for the conjugate factors, when the second term of an equation is absent, and which may sometimes be useful; and adding, in conclusion, a statement of the more interesting inferences which the principles now established suggest.

(32.) If the second term be absent, the general equation of an even degree (1), at page 11, will be

$$x^{2n} + bx^{2n-2} + cx^{2n-3} + dx^{2n-4} + \ldots\ldots + k = 0,$$

which, by the method here employed, may be decomposed into the following conjugate factors, namely,

$$x^{n} + \frac{1}{2} bx^{n-2} \pm \sqrt{\{- cx^{2n-3} + \left(\frac{1}{4} b^2 - d\right) x^{2n-4} - \ldots\ldots - k\}}*$$

If the equation be of the fourth degree,

$$x^4 + bx^2 + cx + d = 0,$$

then the expression X^1, within the brackets, will be

$$- cx + \frac{1}{4} b^2 - d = \frac{1}{4} b^2 - (cx + d),$$

so that, within those limits of x, for which this is negative, no real root of the equation can exist.

The following final example will be illustration sufficient of the above decomposition ; and as I have all along spoken of BUDAN's functions, and BUDAN's rows of signs, I shall exhibit, in this last example, the method of obtaining those rows which I have hitherto tacitly supposed to have been adopted.

(12.) 'MÉMOIRES DE TURIN,' tome vi, p. 171; and LOCKHART's 'Nature of the Roots,' &c., p. 11.

$$x^5 - 5x^2 + 16 = 0,$$

or

$$x^6 - 5x^3 + 16x = 0,$$

$$x^3 - \frac{5}{2} \pm \sqrt{\left(- 16x + \frac{25}{4}\right)}.$$

* It will, of course, be noticed, that this form is preferable to that which would arise from proceeding as at page 11, only inasmuch as we here obtain it at one step, instead of two. If a be made zero in (4), at page 11, the form here given will result, and it is plain that, however many terms after the first may be absent, we may proceed in the same way.

Now the equation proposed may be written

$$x^5 + 0x^4 + 0x^3 - 5\,x^2 + 0x + 16 = 0,$$

furnishing only two variations of sign; hence there are not more than two *real* positive roots.

From X^1 it is plain that $[1, \infty]$ is rejective; so that the two positive roots, if they exist, must lie in the interval $[0, 1]$, and, indeed, in the narrower interval $[0, \cdot 4]$. Transforming them by 1, the operation is as follows:

$$\begin{array}{cccccc}
1 & +\,0 & +\,0 & -\,5 & +\,0 & +\,16 \\
 & 1 & 1 & 1 & -\,4 & -\,4 \\
\cline{2-6}
 & 1 & 1 & -\,4 & -\,4 & + \\
 & 1 & 2 & 3 & -\,1 & \\
\cline{2-5}
 & 2 & 3 & -\,1 & - & \\
 & 1 & 3 & 6 & & \\
\cline{2-4}
 & 3 & 6 & + & & \\
 & 1 & 4 & & & \\
\cline{2-3}
 & 4 & + & & & \\
 & 1 & & & & \\
\cline{2-2}
 & + & & & &
\end{array}$$

As no variations are lost by this transformation, it follows that no roots can lie in the interval $[0, 1]$; and since beyond this, the intervals are wholly rejective, we infer that there are no real positive roots. Hence the equation has but one real root, and that negative.

Inferences from the preceding Investigations.

(33.) (1.) When the signs of X are plus, at the limits of an interval within which a pair of roots is indicated, then, if a root of $X_1 = 0$ lie in a non-rejective part of that interval, the pair of roots indicated are real.

(2.) When the signs of X are minus at those limits, then, if that interval be not wholly non-rejective, the indicated roots are real.

(3.) If the leading coefficient of X^1 under the radical be minus, and of an even degree, and if, preserving this minus sign, the root of $X^1 = 0$, which is nearest to $+\infty$ be a, and the root nearest to $-\infty$, b, then no real roots of $X = 0$ can lie between a and $+\infty$, nor between b and $-\infty$; for, between these limits, X^1 must be negative. Hence the real roots of $X = 0$ are all comprised between those of $X^1 = 0$; so that when these latter are imaginary, all the roots of the proposed equation are imaginary. And, moreover, there cannot be any real root of $X = 0$ between the second root of $X^1 = 0$, reckoning from either extreme root, and the third root; nor between the fourth and fifth; and so on; it being observed, that, in here speaking of the second, third, &c. roots of $X^1 = 0$, we always mean the *real* roots taken in order from either of the extreme roots.

(4.) The reverse happens, should the leading sign of X^1 be plus, that is, the real roots of $X = 0$ can lie only between $[a, \infty]$ and $[b, -\infty]$; and between the second and third; the fourth and fifth roots of $X^1 = 0$; and so on.

(5.) If X^1 be of an odd degree, and the leading coefficient plus, the real roots of $X = 0$ can lie only between $[a, \infty]$, and between the second and third, the fourth and fifth, &c., of the real roots of $X^1 = 0$, reckoning from a. None can lie between b and $-\infty$.

(6.) The reverse of these last circumstances happens if the leading sign of X^1 be minus; no real roots can lie in the interval $[a, \infty]$; they can lie only in the interval $[b, -\infty]$, and between the first and second, the third and fourth, &c., of the real roots of $X^1 = 0$, reckoning from a.

(34.) From the foregoing deductions, we see that, stop at whatever step of the process we may, information respecting

the structure of the equation under examination is always derivable; and although we continue the steps till the operation spontaneously terminates, yet we may take our choice of the X^1 to be finally discussed, that is, we may take it of an odd or of an even degree, as may seem most eligible.

The discussion of X^1, when it does not exceed the fourth degree, may be completely and readily accomplished by STURM's theorem, as I have elsewhere shown;* and X^1 can reach the fourth degree, in the final step of the work, only for equations of the ninth and tenth degree.

If the proposed equation be of the seventh or eighth degree, X^1 in the final step will be a cubic: should this have but one real root, a, we see, from inferences 5 and 6 above, that all the real roots must lie either in the interval $[a, \infty]$ or $[a - \infty]$, according as the leading sign of X^1 is plus or minus.

It is worthy of notice, that whatever X^1 we select for discussion, the other forms of X^1 may often be consulted with advantage; for instance, the X^1 chosen in the analysis of example 7, page 35, shows us that three roots only can lie in the positive region $[0, \infty]$. Knowing this, and finding from the preceding value of X^1, namely,

$$-\frac{3}{4}x^4 + 2x^3 - 2x^2 + x,$$

that the interval $[4, \infty]$ is rejective, as MACLAURIN's limit at once shows, it follows that the three roots adverted to must be confined to the interval $[0, 4]$. A reference to the X^1, furnished at the first step, namely,

$$-x^5 - x^4 + 2x^3 - 2x^2 + x,$$

would supply information still more acceptable; a glance at it is sufficient to show that the interval $[1, \infty]$ is rejective; so that the three roots inquired about are all indicated in the interval $[0, 1]$. We thus see how one form of X^1 may give greater explicitness to the information supplied by another form.

* Analysis and Solution of Cubic and Biquadratic Equations.

I shall add but one observation more:—When the conditions (4), above, have place, then if a, or any number nearer to $+\infty$ than a, be substituted in BUDAN's functions, the variations of sign will show the exact number of imaginary roots indicated between the number substituted and $+\infty$. And if b, or any number nearer to $-\infty$ than b, be substituted, the permanencies of sign will show the exact number of imaginary roots indicated between the number substituted and $-\infty$. No real roots can lie within the limits here mentioned.

SCHOLIA.

(35.) Whatever method be employed for the analysis of a numerical equation, it is always assumed, as an established truth, that every such equation has as many roots as there are units in the exponent of the highest power of the unknown quantity entering it, and which exponent marks the degree of the equation. The equation being reduced to a rational form, there is no question as to the legitimacy of this assumption, in reference to the form thus obtained. But it is now pretty well known that there are irrational equations which have no root; in other words, that irrational functions of x, involving, besides this symbol, only given numbers, may be devised such that no substitution for x, whether real or imaginary, can ever reduce that function to zero. A quadratic function of x may, in general, be split into two conjugate factors, such that one of them shall never vanish, whatever substitution be made for x, while the other shall become zero for each of the two roots of the quadratic of which they are factors. Or the conjugate factors may be such that one root of the resulting quadratic shall satisfy one of them, and the other root the other; while every different substitution shall fail. If, therefore, it be allowable to call an equation, which has only one root, a *simple equation*, we might affirm that certain advanced equations might be decomposed into different sets of simple equations, such that any one in one set shall agree with the corresponding one in

another set only in this, namely, that both of them vanish for a particular root of the equation formed from their product.

Into whatever factors a polynomial be split, it is plain that whatever reduces either of the factors to zero, the others remaining finite must reduce the polynomial itself to zero. When the factors are rational, we know, not only that the *different* roots of these belong to the equation composed of those factors, but that *equal* roots among the former, imply the same equal roots in the latter. The case is different when the factors are conjugate and irrational, as an easy example or two will sufficiently show.

The factors

$$x - a + \sqrt{x - a}$$

$$x - a - \sqrt{x - a}$$

produce the quadratic expression

$$x^2 - (2a + 1)\, x + a\,(a + 1).$$

The first factor becomes zero for $x = a$; the second becomes zero for $x = a$, and $x = a + 1$; and yet the quadratic equated to zero has not the *three* roots, a, a, $a + 1$.

Again, the factors

$$2x - 10 + \sqrt{\{(4 + x)\ (5 - x)\}}$$

$$2x - 10 - \sqrt{\{(4 + x)\ (5 - x)\}}$$

furnish the quadratic equations

$$5x^2 - 41x + 80 = 0.$$

The first factor becomes zero for $x = 5$, so also does the second factor. The first, again, becomes zero for $x = 3\frac{1}{5}$, and yet the quadratic equation, which arises from the product of both, cannot have the *three* roots, 5, 5, and $3\frac{1}{5}$.

(36.) It thus appears that two conjugate factors, such as those here considered, when equated to zero, may seem to furnish three

roots; but it is evidently impossible that they can supply three *different* roots; since, as already remarked, whatever renders a factor zero, must make the product into which it enters, zero; the two roots of the quadratic equation above are 5 and $3\frac{1}{5}$; and no values, besides these, can reduce either of the conjugate factors of its first member to zero. There is room for some observations on the circumstance of one of the roots of the quadratic satisfying *both* the conjugate equations into which it is decomposed; but this is not the place to introduce them; they more properly belong to a subsequent section of the work, the section on zero-symbols.

(37.) At the commencement of these supplementary remarks, it was noticed, that every numerical equation, into which x enters throughout in a rational form, has as many roots, including both real and imaginary, as there are units in the highest exponent of x. This truth has, I believe, hitherto been considered as equally applicable to equations in general; or, at least, it is customary to admit that every equation, numerical or literal, has a root, whether we can actually determine it or not; for it seems to be regarded as a settled point, that CAUCHY has demonstrated this important proposition. But this is a mistake. Whoever carefully examines CAUCHY's proof, will perceive that the main points of the reasoning necessarily involve the assumption that the coefficients of the equation are real or imaginary *numbers*: exclude this condition, and the argument is wholly nugatory.

I do not know whether any one has noticed the dilemma in which the subject would be placed if the demonstrations by ABEL and HAMILTON, that the solution of the general equation of the fifth degree is impossible, be received, and *also* the demonstration by CAUCHY, that every equation has a root, be received. The two conclusions are contradictory. When it is proved that the solution of an equation is impossible, it is proved not merely that we have not the algebraic skill to discover the root, but that the supposed root has really no algebraic existence. This is no more than might be anticipated from *à priori* considerations; every symbolical expression for a root of an equation, with general

symbols for the coefficients, must of necessity be such that, when it is substituted for x, in the proposed equation, the two members of that equation will become absolutely identical, whatever interpretation be given to the symbolical coefficients. The identity spoken of is an identity purely symbolical, irrespective of all arbitrary interpretation, and holds for *every* interpretation, however novel or extravagant. If the general expression for x, in the equation $x^2 + ax = b$, be substituted for x, the identity of the two members remains undisturbed, whatever meanings, however fanciful, we give to a and b. And, in like manner, when we seek for the general solution of an equation of the fifth degree, we presuppose the existence of an expression, involving arbitrary symbols of so very general a character, that when substituted for x, in that equation the two members will become identical for all values or forms that can be imagined of those symbols. It certainly seems more remarkable that such a supposition should be confirmed, in the inferior equations, than that it should prove erroneous in those of more advanced degree.

(38.) What, in one sense, is the perfection of symbolic investigations, is, in another sense, a disadvantage; for it is certainly unfortunate, in the matter in question, that the unavoidable generality of interpretation to which our symbols have claim, really prevents their accomplishing what we exclusively aim at, without, at the same time, accomplishing a great deal more; so that if these superfluous offices prove impossible, those that are actually required, are involved in the condemnation too. A formula for the roots of an equation that should be necessarily restricted to numerical values for the symbols entering it, would not be *algebraical*. Analysis could not furnish such a formula; and this sufficiently accounts for how it happens that the real roots of a numerical equation may be discovered by common arithmetic, when their determination is impossible by algebra. In this point of view, arithmetic has, unquestionably, the advantage over algebra; in the former science we cannot help our symbols being specific in value; in the latter, we cannot help their being general, whenever they are really the symbols of

algebra, and not those of arithmetic. It is not only in the solution of equations that these marked peculiarities operate; other inquiries might be adverted to in which the algebraic forms are impressed with the general mark of impossibility, and from which, therefore, no special real cases can be inferred, though, at the same time, such cases exist in numbers, without any arithmetical restriction at all. The formula which proves the algebraical impossibility of the extension of EULER's theorem, respecting the product of squares, to the case of sixteen squares, is an instance in point; the extension is always arithmetically possible, though always algebraically impossible.

Application of the Conjugate factors to the determination of Two Roots of an Equation, when the others are known.

(39.) The conjugate factors employed in the preceding pages, to facilitate the analysis of an equation, may be applied to many other purposes. Thus, putting $\phi(x)$ for X, in order that we may not confound X, in what follows, with the first member of an equation, and, therefore, writing the factors thus:

$$\phi(x) = [\mathrm{F} + \sqrt{\{\mathrm{F}^2 - \phi(x)\}}] \times [\mathrm{F} - \sqrt{\{\mathrm{F}^2 - \phi(x)\}}] \ldots (1),$$

we see that

$$\log \phi(x) = \log [\mathrm{F} + \sqrt{\{\mathrm{F}^2 - \phi(x)\}}] + \log [\mathrm{F} - \sqrt{\{\mathrm{F}^2 - \phi(x)\}}];$$

or, putting $[\phi(x)]^n$ for $\phi(x)$, and then taking the nth root, the factors may be written

$$\phi(x) = [\mathrm{F} + \sqrt{\{\mathrm{F}^2 - [\phi(x)]^n\}}]^{\frac{1}{n}} \times [\mathrm{F} - \sqrt{\{\mathrm{F}^2 - [\phi(x)]^n\}}]^{\frac{1}{n}}$$

so that

$$\log \phi(x) = \frac{1}{n} \log [\mathrm{F} + \sqrt{\{\mathrm{F}^2 - [\phi(x)]^n\}}] + \frac{1}{n} \log [\mathrm{F} - \sqrt{\{\mathrm{F}^2 - [\phi(x)]^n\}}]$$

where F and n are arbitrary. And these forms for $\log \phi(x)$ would lead to developments different from those usually given. But it would be out of place to dwell upon such applications

here; and I shall therefore confine myself to what is more immediately connected with the doctrine of numerical equations; and shall now show how the conjugate factors (1) may be made available for the purpose of determining expressions for two roots of an equation when all the others have been found.

(40.) First, suppose one root x_1 of the cubic equation

$$x^3 + px^2 + qx + r = 0$$

to be known; then

$$x_1^2 + px_1 + q = -\frac{r}{x_1}$$

will be the product of the two remaining roots.

Let this product be substituted for $\phi(x)$ in the factors (1); then, that those factors may represent the roots themselves, we shall only have to assume F, so that the sum of the factors, namely, 2F, may be the sum of the two roots, that is to say, we must satisfy the condition

$$2\text{F} = -(p + x_1); \quad \therefore \text{F} = -\frac{p + x_1}{2}$$

under which restriction the two factors become

$$-\frac{p + x_1}{2} \pm \sqrt{\left\{\left(\frac{p + x_1}{2}\right)^2 - x_1^2 - px_1 - q\right\}}$$

$$= -\frac{p + x_1}{2} \pm \sqrt{\left\{\left(\frac{p + x_1}{2} - 2x_1\right)\frac{p + x_1}{2} - q\right\}}$$

and these exhibit the values for the two remaining roots of the equation.

Again, suppose two roots, x_1, x_2, of the biquadratic equation,

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

are known, then we shall have

$$x_1^3 + px_1^2 + qx_1 + r = -\frac{s}{x_1}$$

$$x_2^3 + px_2^2 + qx_2 + r = -\frac{s}{x_2}$$

$$\therefore\ (x_1^3 + x_2^3) + p\,(x_1^2 - x_2^2) + q\,(x_1 - x_2)$$

$$= \frac{s}{x_2} - \frac{s}{x_1} = \frac{s\,(x_1 - x_2)}{x_1 x_2}$$

that is, dividing each member by $x_1 - x_2$,

$$(x_1^2 + x_1\,x_2 + x_2^2) + p\,(x_1 + x_2) + q = \frac{s}{x_1 x_2}$$

or, which is the same thing,

$$(x_1 + x_2)^2 + p\,(x_1 + x_2) - x_1\,x_2 + q = \frac{s}{x_1 x_2}$$

the product of the two remaining roots.

As before, let this product be substituted for $\phi(x)$ in the factors (1); then, that those factors may represent the roots themselves, we must assume F, so that 2F may be equal to their sum, that is, we must make

$$2\text{F} = -\,(p + x_1 + x_2)\,; \quad \therefore\ \text{F} = -\frac{p + x_1 + x_2}{2}.$$

Hence the expressions for the two remaining roots are

$$-\frac{p + x_1 + x_2}{2}$$

$$\pm \sqrt{\left\{\left(\frac{p + x_1 + x_2}{2}\right)^2 - (x_1 + x_2)^2 - p\,(x_1 + x_2) + x_1 x_2 - q\right\}}$$

which expression, to give the irrational part of it a form more convenient for actual computation, may be written

$$-\frac{p + x_1 + x_2}{2}$$

$$\pm \sqrt{\left\{\left(\frac{p + x_1 + x_2}{2} - 2\,(x_1 + x_2)\right)\frac{p + x_1 + x_2}{2} + x_1 x_2 - q\right\}}$$

which, we see, reduces to the preceding expression for the roots of a cubic when $x_2 = 0$.

(41.) Expressions having the same object as those above might be obtained differing from these in form. In the 'Analysis of

Cubic and Biquadratic Equations,' the following expression is given for two roots, x_3, x_4, of the biquadratic equation

$$x^4 + qx^2 + r + s = 0,$$

where the term in x^3 is absent, namely,

$$-\frac{x_1 + x_2}{2} \pm \sqrt{\left\{\left(\frac{x_1 - x_2}{2}\right)^2 + \frac{r}{x_1 + x_2}\right\}}$$

which, although less convenient for computation than the corresponding expression derivable from that just given, may deserve a place here for the sake of the following inferences from it, namely:

Supposing r to be positive, which is, of course, always allowable, we see, from the formation of the quantity under the radical sign, that,

1. If two real roots, x_1, x_2, occur in the positive region, the remaining two roots must be real also.

2. If two imaginary roots, x_1, x_2, are indicated in the negative region, the remaining two must be imaginary also. Therefore,

3. If two real roots are detected, without regard to their situation, and if the remaining two are indicated in the negative region, these must be real also. And if two imaginary roots are indicated in either region, and if the remaining two are indicated in the positive region, these must be imaginary also.

(42.) If we were to take an equation of the fifth degree, and to imitate the process above (40), we should obtain three expressions of the third degree, like as the expression (2) of the second degree was obtained; and if, from the double of one of these, the sum of the other two were to be taken, and an obvious division performed, we should get an expression $-\frac{s}{x_1x_2x_3}$, the product of the remaining two roots x_4, x_5, in terms of the three x_1, x_2, x_3, supposed to be already known; and, determining F as before, we should arrive at the values for x_4, x_5, analogous to those above.

But it is not difficult to see that the *product* of two roots of any equation in terms of the other roots is

$$(x_1+x_2+x_3+\ldots)^2+p\,(x_1+x_2+x_3+\ldots)-(x_1x_2+x_1x_3+x_2x_3+\ldots)+q,$$

since all the terms cancel, except that product. Consequently, if this be subtracted from the square of half the *sum* of the same roots, that is, from

$$\left(\frac{p+x_1+x_2+x_3+\ldots}{2}\right)^2$$

the remainder will be the square of half the difference of the same roots; and since the half difference added to and subtracted from the half sum gives the roots themselves, we have generally

$$-\frac{p+x_1+x_2+x_3\ldots}{2}$$

$$\pm\sqrt{\left\{\left(\frac{p+x_1+x_2+x_3+\ldots}{2}\right)^2-(x_1+x_2+x_3+\ldots)^2\right.}$$

$$\left.-p\,(x_2+x_2+x_3+\ldots)+x_1x_2+x_1x_3+x_2x_3+\ldots-q\right\}$$

an expression which may be so modified as to render the irrational part a little more convenient for computation.

Thus in the case of an equation of the fifth degree, instead of the forms

$$-\frac{p+x_1+x_2+x_3}{2}$$

$$\pm\sqrt{\left\{\left(\frac{p+x_1+x_2+x_3}{2}\right)^2-(x_1+x_2+x_3)^2\right.}$$

$$\left.-p\,(x_1+x_2+x_3)+x_1x_2+x_1x_3+x_2x_3-q\right\}$$

we may write

$$-\frac{p+x_1+x_2+x_3}{2}$$

$$\pm\sqrt{\left\{\left(\frac{p+x_1+x_2+x_3}{2}-(x_1+x_2+x_3)\right)\frac{p+x_1+x_2+x_3}{2}\right.}$$

$$\left.+x_1(x_2+x_3)+x_2+x_3-q\right\}.$$

It thus appears that the forms previously deduced might have been arrived at independently of the conjugate factors; but it

was the previous introduction of those factors that suggested the general considerations here employed.

(43.) In reference to the formulæ here established, it may not be superfluous to observe, that they will be found more especially useful in those cases in which all the roots but two are real; as they will enable us to exhibit the imaginary pair, by aid of the real roots, with comparatively little expense of calculation. And even when all the roots are real, a saving of figures is, in general, still effected by them. But, in comparing formulæ of this kind with the numerical process of HORNER, it must always be remembered that HORNER's method supplies the roots in an explicit form; whereas, in expressions for them, such as these, there yet remains an unperformed operation, indicated by the radical; which operation, however, in the case of a pair of imaginary roots is, of course, impracticable, and therefore nothing further can be done. But in all formulæ for imaginary roots, into which *approximate values* only of the real roots enter, it is necessary, in delicate cases,—that is, in those cases in which a very slight change in any of the coefficients would convert real roots into imaginary, or imaginary roots into real,—it is necessary, in such cases, to push these approximations to a more than usual extent, in order to avoid the risk of converting imaginary roots into real, and *vice versâ;* for there is no hope of attaining the imaginary forms *accurately,* when we employ approximations only to the real quantities which enter into our expressions for them; and, in such critical cases as those just alluded to, the error in the approximations may be sufficiently great to lead to a false conclusion as to the character of the roots.

It appears to me, however, and I have elsewhere expressed the same opinion, that we may always safely regard a pair of roots to be real and equal, provided they would actually become so upon making changes, in the coefficients of the proposed equation, so minute as to render the difference between the two states of that equation of no practical moment. Whether the roots converted into equality by such changes be originally real or imaginary, should not form any item of consideration. There is no

doubt that real roots and imaginary roots are distinct classes, marked by distinct peculiarities; but it may be questioned, whether pairs belonging to the latter class are not sometimes too hastily rejected as mere symbols of *impossibility*. If a pair of real roots, kept together for seven or eight places of decimals, and then separated—a very possible case—the error committed by rejecting the non-concurring figures, and pronouncing the two roots to be equal, would not be considered as of serious consequence; yet it is *impossible* that these equal roots can strictly belong to the equation, for if either of them be substituted in the equation, the left-hand member of it, instead of becoming zero, would be a minute finite value; that is, the equal roots really belong to the equation only after a minute change has been made in one or more of the coefficients; if a similar change to that which thus converts unequal roots into equal roots, should convert imaginary roots into equal roots, it is plain that, whatever considerations justify the change in the one case must equally justify it in the other; in both cases, a pair of equal roots would belong to an equation differing from that proposed by a quantity too small to deserve notice. But, for further observations on this point, reference may be made to 'The Theory and Solution of Algebraical Equations,' pp. 307-12.

(44.) As a conclusion to this article, it may be noticed, that any function $\phi(x)$ may be decomposed into as many pairs of conjugate factors as we please; thus it is evident that

$$\phi(x)=[F+\sqrt{\{F^2-ff_1f_2\cdots f_n\cdot\phi(x)\}}]\times[F^2-\sqrt{\{F^2-ff_1f_2\cdots f_n\cdot\phi(x)\}}]$$

$$\times\left[F_1+\sqrt{\left\{F_1^2-\frac{1}{f}\right\}}\right]\times\left[F_1-\sqrt{\left\{F_1^2-\frac{1}{f}\right\}}\right]$$

$$\times\left[F_2+\sqrt{\left\{F_2^2-\frac{1}{f_1}\right\}}\right]\times\left[F_2-\sqrt{\left\{F_2^2-\frac{1}{f_2}\right\}}\right]$$

$$\vdots$$

$$\times F_{n-1}+\sqrt{\left\{F^2_{n-1}-\frac{1}{f_n}\right\}}\Bigg]\times\left[F_{n-1}-\sqrt{\left\{F^2_{n-1}-\frac{1}{f_n}\right\}}\right]$$

where F, F_1, F_2, &c., and f, f_1, f_2, &c. are arbitrary.

I should not have inserted this obvious identity but for the fact, that there is no foreseeing what even the plainest truisms of analysis may hereafter suggest; the simplest case of this general form, considered in the preceding pages, leads to results which, I venture to think, will be deemed not unworthy of notice in any future account of the progress of the subject to which the present essay is chiefly devoted.

(45.) One of the most simple applications of the formulæ now given is to the general solution of equations of the fourth degree. For since

$$\phi(x)=[F+\sqrt{\{F^2-f\phi(x)\}}]\times[F-\sqrt{\{F^2-f\phi(x)\}}]\dots(1)$$

$$\times\left[F_1+\sqrt{\left\{F_1^2-\frac{1}{f}\right\}}\right]\times\left[F_1^2-\sqrt{\left\{F_1^2-\frac{1}{f}\right\}}\right]\dots(2)$$

it follows, that if we have the biquadratic equation

$$x^4+px^3+qx^2+rx+s=0\dots\dots\dots\dots(3)$$

and that we put s for $\phi(x)$ in the formula, the four factors (1) (2) may be regarded as the conjugate factors of the four roots of the equation. The first pair (1) of these roots will obviously be furnished by the quadratic

$$x^2-2Fx+fs=0$$

and the second pair (2) by the quadratic

$$x^2-2F_1x+\frac{1}{f}=0$$

so that if we multiply these two quadratic expressions together, and equate the resulting coefficients with those of the like powers of x in (3), we shall get the requisite equations of condition for the solution of the equation proposed; if p be zero, the quadratic factors will, of course, be

$$(x^2-2Fx+fs)\left(x^2+2Fx+\frac{1}{f}\right)$$

since, in this case, $F + F_1$ must be zero: we shall thus get the solution of DESCARTES.

It is easy to see how the quadratic factors of any equation of an even degree may, in like manner, be symbolically expressed.

While thus noticing one form of investigation for the solution of the general biquadratic equation, it may not be out of place to advert to another form immediately suggested by the mode of decomposing a function employed in the preceding pages.

Resuming the equation (3), the conjugate factors of the first member admit of the introduction of an indeterminate quantity u, thus

$$x^2 + \frac{px}{2} \pm \sqrt{\left\{\left(\frac{p^2}{4} - q\right) x^2 \, rx - s\right\}}$$

$$x^2 + \frac{px}{2} + u \pm \sqrt{\left\{\left(\frac{p^2}{4} - q + 2u\right)x^2 + (pu - r)\,x + u^2 - s\right\}}.$$

And if u be now determined so as to make

$$4\left(\frac{p^2}{4} - q + 2u\right)(u^2 - s) = (pu - r)^2$$

the expression under the radical will be a complete square of the form $(mx + n)^2$, where m and n are known. Hence the biquadratic becomes decomposed into the two quadratics

$$x^2 + \frac{px}{2} + u + (mx + n) = 0,$$

$$x^2 + \frac{p}{2} + u - (mx + n) = 0,$$

and thus the solution, when p is zero, becomes that of LOUIS FERRARI. And this mode of arriving at it is certainly more direct and easy than any method hitherto proposed. The steps might be imitated for an equation of the sixth degree; but to obtain, as here, an equation of condition of the third degree, or even of the fourth, would necessitate a determinate relation among the coefficients.

On some Criteria of Imaginary Roots analogous to the Criteria of Newton.

(46.) In a former publication,* I have discussed some useful formulæ for discovering imaginary roots in a numerical equation, from inspecting the coefficients of its terms. The new forms now proposed will, I think, prove an acceptable addition to those here adverted to. They are not only of greater simplicity, but they will often succeed in detecting the presence of imaginary roots in cases where NEWTON's formulæ — the basis of those before given—would prove inefficient.

These new forms are deduced as follows:

Let the equation

$$A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + A_{n-3} x^{n-3} + \ldots . A_1 x + A_0 = 0$$

be multiplied by $x - a$; that is, let a new undetermined real root a be introduced into the equation.

Then whatever imaginary roots are indicated in the new equation, the same, of course, enter into the original.

This new equation is

$$A_n x^{n+1} + (A_{n-1} - a\,A_n)\,x^n + (A_{n-2} - a\,A_{n-1})\,x^{n-1} +$$

$$(A_{n-3} - a\,A_{n-2})\,x^{n-2} + \ldots . (A_0 - a\,A_1)\,x + a\,A = 0,$$

in which a is entirely arbitrary; and may therefore be made to satisfy any condition we please. It may, for instance, be determined so as to render any one of these compound coefficients zero. And if, in conjunction with this determination of a, the original coefficients be so related as to cause the compound coefficients, on each side of this zero, to be of like signs, we shall at once recognise the presence of imaginary roots in the proposed equation.

Equating then the several coefficients to zero, one after the

* Researches respecting the Imaginary Roots of Equations, forming an Appendix to the Treatise on the Theory of Equations.

other, commencing with the second, and determining the adjacent pair of coefficients in each case conformably to this condition, we shall have the following sets of relations, the existence of any one of which will imply the entrance of a pair of imaginary roots into the proposed equation.

$$A_n = +,\ A_{n-1} - a A_n = 0,\ A_n A_{n-2} - A^2_{n-1} = +$$

$$A^2_{n-1} - A_n A_{n-2} = +,\quad A_{n-2} - a A_{n-1} = 0,\ A_{n-1} A_{n-3} - A^2_{n-2} = +$$

$$A_{n-2} - A_{n-1} A_{n-2} = +,\ A_{n-3} - a A_{n-2} = 0,\ A_{n-2} A_{n-4} - A^2_{n-3} = +$$

&c. &c.

And, therefore, a being always assumed so as to satisfy one of the midde equations, those on each side will furnish the following series of criteria of imaginary roots, namely,

$$A_n = +,\ A_n A_{n-2} > A^2_{n-1} \ldots\ldots\ldots\ldots (1)$$

$$A^2_{n-1} > A_n A_{n-2},\ A_{n-1} A_{n-3} > A^2_{n-2} \ldots\ldots\ldots (2)$$

$$A^2_{n-2} > A_{n-1} A_{n-3},\ A_{n-2} A_{n-4} > A^2_{n-3} \ldots\ldots (3)$$

&c. &c.

Now, assuming, according to custom, that A_n is always plus, it is plain that if any one of the right-hand conditions have place, without regarding those on the left, a pair of imaginary roots will necessarily be indicated; because, although the accompanying left-hand condition should not exist, yet, by ascending to the preceding pair of conditions, and thence to that next in order, and so on, we should evidently, at length, arrive at a coexistent pair. It is equally evident that no two consecutive pairs can exist simultaneously; since the second condition, in any one pair, is opposed to the first in the pair next following.

Hence, although all the right-hand conditions marked (1), (2), (3), &c., were found to have place, yet more than one pair of imaginary roots could not be safely inferred; these conditions, therefore, ought not to be regarded as anything more than so many concurrent indications of the same thing.

Whenever this concurrence ceases, by a failure of one of the right-hand conditions referred to, or, which is the same thing,

by a fulfilment of the left-hand condition next in order, then preparation is made for a new indication, totally distinct from the indications that have preceded. And if such new indication offer itself, a distinct pair of imaginary roots may be inferred.

Hence the series of conditions (1), (2), (3), &c., furnish us with criteria of imaginary roots somewhat analogous to those of NEWTON; much simpler, however, in form, but to be employed in exactly the same manner. I shall give two examples of this application:

1st. $5x^8-2x^7+3x^6-24x^5-16x^4+x^3-4x^2-2x-60=0.$

Applying the criteria to this equation, we discover the existence of six imaginary roots, the same as by the rule of NEWTON. ('Researches,' &c., page 47.)

2d. $9x^5-5x^4+4x^3-3x^2+6x+A_0=0.$

In this example, NEWTON's rule detects only a single pair of imaginary roots; the criteria above discover *two* pairs; so that the equation has but one real root.

It is scarcely necessary to add that, as in the rule of NEWTON, though the sign $>$ be replaced by $=$ the criteria still hold.

(47.) It will be easy to give to these criteria a more general form; it will only be necessary to multiply the several coefficients of the equation, in the manner so often employed by NEWTON and MACLAURIN, by the terms of an arithmetical progression, such as 1, 2, 3, &c., or, more generally, k, $k+1$, $k+2$, &c., and to replace the original coefficients by the results; so that, including the criteria already established, we thus obtain the three following sets, namely,

$$\begin{array}{c}
A_n A_{n-1} > A_{n-1}^2 \\
A_{n-1} A_{n-3} > A_{n-2}^2 \\
A_{n-2} A_{n-4} > A_{n-3}^2 \\
\vdots \\
A_2 A_0 > A_1^2
\end{array}$$

$$3\, A_n\, A_{n-2} > 4\, A^2_{n-1}$$

$$8\, A_{n-1}\, A_{n-3} > 9\, A^2_{n-2}$$

$$15\, A_{n-2}\, A_{n-4} > 16\, A^2_{n-3}$$

$$\vdots$$

$$(n^2-1) A_2\, A_0 > n^2\, A_1{}^2$$

$$k\,(k+2)\, A_n\, A_{n-2} > (k+1)^2\, A^2_{n-1}$$

$$(k+1)\,(k+3)\, A_{n-1}\, A_{n-3} > (k+2)^2\, A^2_{n-2}$$

$$(k+2)\,(k+4)\, A_{n-2}\, A_{n-4} > (k+3)^2\, A^2_{n-3}$$

$$\vdots$$

$$(k+n-2)\,(k+n)\, A_2\, A_0 > (k+n-1)^2\, A_1{}^2.$$

And in these criteria we may, if we please, write A_0, A_1, A_2, &c., in the place of A_n, A_{n-1}, A_{n-2}, &c., and *vice versâ*.

It will be observed, that the two latter sets of criteria are really distinct from the first set, and are not comprehended in that set; for although, whenever any one of these holds, the corresponding one in the first set must hold, also; yet one of these may fail without any failure in the analogous form of the first set.

The last general forms, when expressed in words, furnish the following rule, which it will not be difficult to remember:

Rule.—Commencing with the second coefficient, and proceeding towards the right, or with the last but one, and proceeding towards the left, multiply the successive coefficients by any series of consecutive whole numbers, and square the results; the square of each coefficient will thus be multiplied by an integral square. Let the product of the adjacent coefficients on each side be now multiplied by the same quantity, *minus* 1, and compare these results with the former, as in Newton's rule.

I here terminate the essay on the Analysis of Equations by the aid of Conjugate Factors. It will be readily seen that this application of those factors is only one out of many inquiries into which they may be advantageously introduced. In the theory of curves and surfaces, for instance, the employment of the conjugate factors will often be found of use; indeed, any facilities in the analysis of a general equation must be accompanied with like facilities in the discussion of a curve or a surface.

I think, too, that in the investigation of certain trigonometrical formulæ, the decomposition adverted to will afford acceptable aid. In these, as in nearly all the general theorems of analysis, nothing more is expressed but identity of value under diversity of form. In fact, the more advanced parts of analysis are almost exclusively devoted to transformations of this kind. Such being the case, any addition to our existing analytical forms, however simple and obvious it may appear, when actually exhibited, is not without a certain amount of value in the estimation of the algebraist, who well knows that, to such simple, and perhaps almost axiomatic operations upon the previously-existing expressions, the most important steps, both in pure and applied science, are traceable. I need mention here, in corroboration of this, only the trigonometrical transformations of EULER, and the dynamical principle of D'ALEMBERT.

For some additional remarks on the subject of the present Essay, see Note (A), at the end of the volume.

NAMES OF SUBSCRIBERS.

James Apjohn, M.D. &c., Professor of Applied Chemistry, Trin. Col., Dublin.
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Professor Finlay, M.A., Manchester College.
Wm. Finlay, Esq., First Math. Master, Middle School, Liverpool Collegiate Inst.
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